DAILY UPDATE FOR MATH 147 FALL 2025

Monday, August 18. This class was devoted to an informal discussion of the topics we plan to cover this semester. We began by describing the type of functions we will study, namely scalar valued and vector valued functions of several variables. We illustrated, in a general way, how with the proper definitions, many of the basic concepts from Calculus I and Calculus II can be extended to functions of several variables. On the other hand, we noted, for example, that for a function of two variables, one can measure a rate of change in infinitely many different directions, something not encountered in Calculus I.

We then gave a brief overview of how the integration process works in general: One always has a function to integrate (the *integrand*) and a domain of integration. We then described how the integration process works in all scenarios we will encounter during the semester. Namely, starting with a domain of integration and a function defined on that domain, we proceed as follows:

(i) Subdivide the domain of integration into small portions of a similar type, e.g, if the domain of integration is a solid, subdivide into smaller solids; if the domain of integration is a curve, subdivide into smaller curves. One can assume the individual portions have the same size.
(ii) Choose a point in each subdivision and evaluate the function at that point.
(iii) Multiply the answer in (ii) by the size of the subdivision, e.g., volume if a solid, length if a curve.
(iv) Add the quantities in (iii).
(v) Take the limit of the sums in (iv) as the size of the subdivisions tend to zero.

The resulting numerical value depends only on the function and the underlying geometry of the domain of integration. We noted that the real challenge is to calculate this quantity.

We continued our discussion of functions of several variables, with the examples (like, though not exactly) $f(x,y) = x^2 + y^2$, $g(x,y,z) = \frac{1}{x^2+y^2+z^2}$, and $h(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$. We noted that $f(-2,3) = (-2)^2 + 3^2 = 13$, $g(1,2,3) = \frac{1}{1^2+2^2+3^2} = \frac{1}{14}$, and $h(1,2,\ldots,n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

We then noted that the range and domain for functions of several variables, have the same meaning as for functions of a single variable, namely the domain is the set of allowable inputs and the range is the set of possible outputs. In the case of a function of two variables, say, the allowable inputs will elements of $\mathbb{R}^2$, but the the outputs are always real numbers. For the functions defined above, we have:

(i) $f(x,y)$: Domain $= \mathbb{R}^2$ and range $= \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$.
(ii) $g(x,y,z)$: Domain $\mathbb{R}^3 \backslash (0,0,0)$ and range $= \{\alpha \in \mathbb{R} \mid \alpha > 0\}$.
(iii) $h(x_1, x_2, \ldots, x_n)$: Domain $= \mathbb{R}^n$ and range $= \mathbb{R}$.

One notes that for the function $t(x,y) = \sqrt{1 - x^2 - y^2}$, the domain is the the unit circle of radius one centered at the origin in $\mathbb{R}^2$ and its interior, while the range is $\{\alpha \in \mathbb{R} \mid 0 \leq \alpha \leq 1\}$. After this we discussed graphing functions of two variables $z = f(x,y)$ and how the level curves $f(x,y) = c$ for different values of c help to understand the graph of $f(x,y)$. We also cautioned that the level curves do not give a complete picture of the graph, since for example, the level curves of $f(x,y) = x^2 + y^2$ and $g(x,y) = \sqrt{x^2 + y^2}$ are circles of increasing radii, as $c > 0$ increases, but the graph of $f(x,y)$ is a *paraboloid* (see here) while the graph of $g(x,y)$ is a *cone* (see here). We noted that the difference between these two surfaces can be seen by taking curves obtained by setting $y = c$, and in particular $y = 0$. In this cross section, the graph of $f(x,y)$ is a parabola while the graph of $g(x,y)$ is the graph of $z = |x|$ in the xz-plane, i.e., two rays emanating from the origin at 45 degrees.

Wednesday, August 20. We began a discussion of limits and continuity for functions of several variables. As a reminder, we first discussed the concepts of limit and continuity, for a function of one variable. Just as $\lim_{x\to a} f(x) = L$ intuitively mean that the values of $f(x)$ approach L as x approaches x, it should be the case that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that the values of $f(x,y)$ approach the real number L, as (x,y) approaches (a,b). We noted that while limits for functions of one variable involve a approaching x from either the right or left, for limits with functions of two variables, there are infinitely many ways (x, y) can approach (a, b). On the other hand, using the notion of distance, the definition of a limit still takes a very similar form as in one variable: Namely, we say $\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$, such that $|(x, y) - (a, b)| < \delta$ implies $|x - L| < \epsilon$. We then defined $f(x, y)$ to be continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Before calculating some examples, we noted that the following rules for taking limits hold, and that these are the same rules that apply when taking limits of function of one variable.

Theorem 12.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\lim_{(x,y)\to(x_0,y_0)} x = x_0$; $\lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

Thus, for example,

$$\begin{aligned}
\lim_{(x,y)\to(1,2)} \frac{3x^2y + 4xy}{x + y} &= \frac{\lim_{(x,y)\to(1,2)} 3x^2 + 4xy}{\lim_{(x,y)\to(1,2)} x + y} \\
&= \frac{3\lim_{(x,y)\to(1,2)} x^2 + 4\lim_{(x,y)\to(1,2)} xy}{\lim_{(x,y)\to(1,2)} x + \lim_{)x,y)\to(1,2)} y} \\
&= \frac{3(\lim_{x\to1} x)^2 + 4(\lim_{x\to1} x)(\lim_{y\to2} y)}{\lim_{x\to1} x + \lim_{y\to2} y} \\
&= \frac{3 \cdot 1^2 + 4 \cdot 1 \cdot 2}{1 + 3} \\
&= \frac{11}{3}.
\end{aligned}$$

We also noted that when applying the rules above to $\lim_{(x,y)\to(a,b)} f(x, y)$ it is crucial to use limits, and not merely substitute (a, b) for (x, y). For example, let

$$a(x) = \begin{cases} 1, & \text{if } x \neq 3 \\ 2, & \text{if } x = 3, \end{cases}$$

so that $\lim_{x\to3} a(x) = 1, \neq 2$ and $b(y) = 2y$. Then for $f(x, y) = a(x)b(y)$, we have $\lim_{(x,y)\to(3,2} f(x, y) = 1 \cdot 4 = 4$.

We then noted that the same rules above for limits apply for continuity, e.g., sum, products etc of functions continuous at (a, b) are continuous at (a, b). We finished class by first showing $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, for $f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0 \text{ if } (x, y) = (0, 0). \end{cases}$ and discussing an approach for evaluating the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ for $f(x,y) = \begin{cases} \frac{3x^2y-y^3}{x^2+y^2}, \text{ if } (x,y)\neq(0,0) \\ 0 \text{ if } (x,y)=(0,0) \end{cases}$ using trigonometric substitution with $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Friday, August 22. We reviewed the definitions of limit and continuity for scalar functions of the form $f(x,y)$. After doing an example where the function was continuous so that the limit was obtained by substitution, we considered the following limit: $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$. We noted that the limit was zero as (x,y) approached (0,0) along any line $y = kx$, for $k \in \mathbb{R}^2$, but that the limit is not zero as (x,y) approached (0,0) along the curve (y^2,y). This shows that for $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, it is not just a matter of the limit existing as (x,y) approached (a,b) from all possible directions.

We then worked through an ϵ, δ proof of the two facts: (i) $\lim_{(x,y)\to(a,b)} x = a$ and (ii) The limit of a sum is the sum of the limits.

We ended class by discussing limits and continuity for vector valued functions $F : \mathbb{R}^n \to \mathbb{R}^m$, noting that the concept of distance renders the corresponding definitions almost identical to the case of $f(x,y)$ previously discussed: For $P \in \mathbb{R}^n$ and $L \in \mathbb{R}^m$, writing $\underline{x}$ for $(x_1,\dots,x_n)$, we have $\lim_{\underline{x}\to P} F(\underline{x}) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $||F(\underline{x}) - L|| < \epsilon$, whenever $||(\underline{x}) = L|| < \delta$ and, moreover, $F(\underline{x})$ is continuous at P if $\lim_{\underline{x}\to P} F(\underline{x}) = F(P)$. We then stated the theorem that if

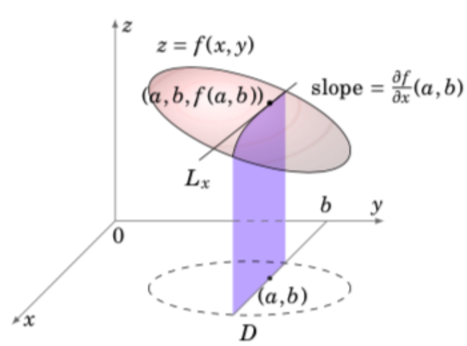
$$F(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)),$$

then writing $\underline{x}$ for $(x_1,\dots,x_n)$, and taking $P \in \mathbb{R}^n$, $L = (l_1,\dots,l_m) \in \mathbb{R}^m$, $\lim_{\underline{x}\to P} F(\underline{x}) = L$ if and only if $\lim_{\underline{x}\to P} f_j(\underline{x}) = l_j$, for all $1 \le j \le m$, and $F(\underline{x})$ is continuous at P if and only if each $f_j(\underline{x})$ is continuous at P, i.e., $\lim_{\underline{x}\to P} f_j(\underline{x}) = l_j$, for all $1 \le j \le m$. We discussed why this is intuitively true in the case of $F : \mathbb{R}^2 \to \mathbb{R}^2$.

Monday, August 25. We began our discussion of partial derivatives. We noted that given a function $f(x,y)$, if we start at the point (a,b), then there are infinitely many directions moving away from (a,b) for which we could seek the rate of $f(x,y)$. Our analysis began with the rate of change of $f(x,y)$ at (a,b) in a direction parallel to the x-axis. We noted that this can be analyzed by intersecting the graph of $z = f(x,y)$ with the plane $y = b$. We observed that doing so reduces the problem to calculating the derivative of the function $c(x) := f(x,b)$ at $x = a$. Thus,

$$c'(a) = \lim_{h\to 0} \frac{c(a+h)-c(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}.$$

If this limit exists, this is the *partial derivative* of $f(x,y)$ with respect to x at (a,b), which we denote by $\frac{\partial f}{\partial x}(a,b)$ or $f_x(a,b)$. Geometrically, we can think of $\frac{\partial f}{\partial x}(a,b)$ as the slope of the line passing through the point $(a,b,f(a,b))$ tangent to the curve $z = f(x,b)$ lying on the graph of $z = f(x,y)$.



(a) Tangent line L_x in the plane $y = b$

We then used the limit definition to calculate $f_x(1,1)$ for the function $f(x,y) = 2x+3y$ and $g_x(2,-1)$ for the function $g(x,y) = x^3y + 3xy^2$. The resulting values were 2 and -9. We then repeated the limit calculations to find the general values of $f_x(x,y)$ and $g_x(x,y)$, and observed that $f_x(x,y) = 2$ and $g_x(x,y) = 3x^2y + 3y^2$. This suggests that in general, when the relevant limits exist, the partial derivative of an arbitrary $f(x,y)$

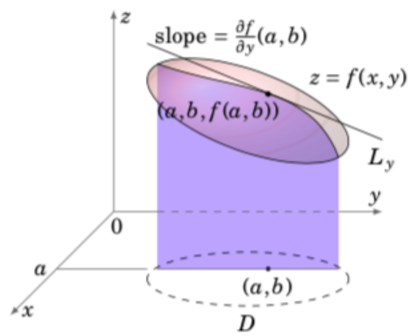
with respect to x is obtained by differentiating the expression for $f(x, y)$ with respect to x, treating y as a constant. We then did this direct calculation for $h(x, y) = 6x^2y^3e^{x^2+2y^2} + 5\cos(xy)$, obtaining

$$\frac{\partial h}{\partial x} = 12xy^3e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(2x) - 5\sin(xy) \cdot y.$$

We then noted that the discussion above applies equally well to the variable y, so that one defines the partial derivative of $f(x, y)$ with respect to y at (a, b) as

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h},$$

provided the limits exists. This then represents the slope of the line tangent tangent to the graph $z = f(x, y)$ at (a, b) along the curve obtained by intersecting the graph of $z = f(x, y)$ with the plane $x = a$.



(b) Tangent line L_y in the plane $x = a$

Accordingly, the function $\frac{\partial f}{\partial y}$ is calculated by differentiating $f(x, y)$ with respect to y, treating x as a constant. Doing so for $h(x, y)$ above yields,

$$\frac{\partial h}{\partial y} = 18x^2y^2e^{x^2+2y^2} + 6x^2y^3e^{x^2+2y^2}(4y) - 5\sin(xy) \cdot x.$$

We ended class by noting that the following rules for partial differentiation (and their more general counter parts) follow from the corresponding familiar rules from Calculus 1.

For functions $f(x, y)$, $g(x, y)$, $h(t)$:

(i) $(f + g)_x = f_x + g_x$
(ii) $(fg)_x = f_x g + f g_x$
(iii) $(h(f(x, y))_x = h'(f(x, y)) \cdot f_x(x, y)$,

and similarly for partial with respect to y.

Wednesday, August 27. We began class reaclling the partial derivatives of functions of two variables and noted that the definition easily carries over to functions of several variables, the general case being a function $f(x_1, x_2, \ldots, x_n)$. In this case, we defined the partial derivative of $f(x_1, \ldots, x_n)$ with respect to x_i at $(a_1, \ldots, a_n)$ to be

$$\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)}{h},$$

if the limit exists. We then noted that to calculate partial derivatives of functions of many variables with respect to a given variable, we differentiate with respect to that variable, treating all remaining variables as constants. We then calculated the partial derivatives of the function $f(x, y, z, w) = x^2y^3z^4w^5 + e^{xy+zw}$.

We then began a discussion of finding the tangent lines to the graph of $z = f(x, y)$ in the x and y directions at the point $(a, b, f(a, b))$, using the fact that for the function $f(x, y)$, if $f_x(a, b)$ exists, $f_x(a, b)$ as the slope of the line passing through the point $(a, b, f(a, b))$ tangent to the curve $z = f(x, b)$ lying on the graph of $z = f(x, y)$ in the plane $y = b$. From this, were able to derive the parametric equation for this line:

$$L_x(t) = (a, b, f(a, b)) + t \cdot (1, 0, f_x(a, b)).$$

Similarly, we noted that if $f_y(a,b)$ exists, $L_y(t) = (a,b,f(a,b)) + t\cdot(0,1,f_y(a,b))$ gives a tangent line in the y direction.

We then noted that having good tangent lines in two directions leads to the expectation of having a tangent plane at the point $(a,b,f(a,b))$. We ended class by recalling the fact that if $P=(a,b,c)$ is a point on the plane F and $\vec{n}=(n_1,n_2,n_3)$ is normal to F, then the equation of F is the set of all points in $\mathbb{R}^3$ satisfying $0 = n_1(x-a)+n_2(x-b)+n_3(z-c)$.

Friday, August 29. We began class by recalling the parametric equations of the tangent lines in the x and y directions to the graph of $z=f(x,y)$ at the point $(a,b,f(a,b))$. We then noted that having good tangent lines in two directions leads to the expectation of having a tangent plane at the point $(a,b,f(a,b))$. After recalling the fact that a plane is determined by a point and a normal vector, we arrived at the equation for what *should be* the relevant tangent plane:

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b).$$

We noted that geometrically, the cone $z=\sqrt{x^2+y^2}$ clearly does not have a tangent plane at (0,0,0), and this was confirmed by verifying that neither partial derivative exists at (0,0). We then pointed out that the situation is more subtle than this, by considering the function $f(x,y) = ||x|-|y|| - |x| - |y|$. In this case, $f_x(0,0)$ and $f_y(0,0)$ exist, and equal 0, so that $z=0$ is the expected tangent plane. We then noted that there is no tangent plane in this case, because the curve $z=f(x,2x)$ on the graph of $f(x,y)$ does not have a tangent line at (0,0,0). **Thus, we emphasized that the existence of partial derivatives does not guarantee the existence of a tangent plane.** We then noted that this difficulty is taken care of in the same way as in Calculus I by using the concept of differentiability. We then stated the following:

Definition. Given $f(x,y)$ and (a,b) in the domain of $f(x,y)$. Suppose that $f_x(a,b)$ and $f_y(a,b)$ exist and set $L(x,y) := f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$. Then $f(x,y)$ is *differentiable* at (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{||(x,y)-(a,b)||} = 0.$$

We observed that the numerator and denominator in the limit above both go to zero, so that if the quotient goes to zero, $f(x,y)-L(x,y)$ is going to zero significantly faster than the denominator. Thus, we concluded that if $f(x,y)$ is differentiable at (a,b), then:

(i) The function $L(x,y)$ is a good linear approximation to $f(x,y)$ for points (x,y) sufficiently close to (a,b).
(ii) The proposed tangent plane $z=L(x,y)$ is, in fact, tangent to the graph of $f(x,y)$ at (a,b).

We finished class by showing directly that $f(x,y)=xy$ is differentiable at (a,b), for all $(a,b)\in\mathbb{R}^2$.

Wednesday, September 3. We began class by reviewing the definition given in the previous lecture regarding the differentiability of $f(x,y)$ at (a,b), noting $L(x,y)$, as previously defined, is a good linear approximation to $f(x,y)$ at (a,b). Geometrically, this means that there is a well defined tangent plane to the graph of $z=f(x,y)$ at $(a,b,f(a,b))$. In other words, does the proposed tangent plane

$$z = L(x,y) := f_x(a,b)(x-a)+f_y(a,b)(y-b)+f(a,b)$$

does a good job of approximating $f(x,y)$ in the vicinity of (a,b).

We then stated the following **very important** theorem:

Theorem. (Differentiability Criterion) Given $f(x,y)$ and (a,b) in the domain of $f(x,y)$ and suppose $f(x,y)$ has the property that $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous in an open disk about (a,b). Then $f(x,y)$ is differentiable at all points in that disk, including the point (a,b). Here an open disk of radius r about (a,b) means the set of points in $\mathbb{R}^2$ whose distance from (a,b) is less than r.

To illustrate the connection between the conditions in the theorem, we considered two examples. First we looked at $f(x,y)=||x|-|y||-|x|-|y|$ and then $f(x,y)=\frac{2xy}{\sqrt{x^2+y^2}}$. In both cases we saw that both partials exist at (0,0), but that the function is not differentiable at (0,0). In each case the failure of differentiability was due to the fact that the partial derivatives are not continuous at (0,0).

We then noted that if $f(x,y)$ is differentiable at (a,b) and Δx and Δy are small, then

$$\begin{aligned} f(a+\Delta x, b+\Delta y) &\cong L(a+\Delta x, b+\Delta y) \\ &= f_x(x,b)(a+\Delta x - a) + f_y(a,b)(b+\Delta y - b) + f(a,b) \\ &= f_x(a,b)\Delta x + f_y(a,b)\Delta y + f(a,b). \end{aligned}$$

We then illustrated this with the following example.

Example. Use the linear approximation to estimate $f(1.01, 2.02)$, for $f(x,y) = x^2 + 2xy$. We assume that $f(x,y)$ is differentiable at $(1,2)$. We calculated $f_x(x,y) = 2x + 2y$, so $f_x(1,2) = 6$; $f_y(x,y) = 2x$, so $f_y(1,2) = 2$. We also have $\Delta z = .01$ and $\Delta y = .02$. Thus,

$$f(1.02, 2.02) \cong f_x(1,2)(.01) + f_y(12)(.02) + f(1,2) = 6(.01) + 2(.02) + 5 = 5.1.$$

Note that $f(1.01, 2.02) = (1.01)^2 + 2(1.01)(2.02) = 5.1005$, so the approximation 5.1 is a good one.

We then recorded the facts : If $f(x,y)$ and $g(x,y)$ are differentiable at (a,b), then so are:

(i) $f(x,y) + g(x,y)$, $f(x,y)g(x,y)$, $\frac{f(x,y)}{g(x,y)}$, if $g(a,b) \neq 0$.
(ii) $h(f(x,y))$, if $h(t)$ is differentiable at $f(a,b)$.
(iii) Moreover, $f(x,y)$ and $g(x,y)$ are continuous at (a,b).

We ended class by discussing how to extend the concept of differentiability to functions of more than two variable by taking $f(x_1, \ldots, x_n)$ and $P = (a_1, \ldots, a_n) \in \mathbb{R}^n$. First assume that each partial derivative $\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n) = \frac{\partial f}{\partial x_i}(P)$ exists, so that we can form the linear function

$$L(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x - a_i) + f(P).$$

Then $f(x_1, \ldots, x_n)$ is differentiable at P if

$$\lim_{(x_1,\ldots,x_n)\to P} \frac{f(x_1,\ldots,x_n) - L(x_1,\ldots,x_n)}{||(x_1,\ldots,x_n) - P||} = \lim_{(x_1,\ldots,x_n)\to P} \frac{f(x_1,\ldots,x_n) - L(x_1,\ldots,x_n)}{\sqrt{x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}} = 0.$$

One then has essentially the same criterion for differentiability, namely, $f(x_1, \ldots, x_n)$ is differentiable at P if the partial derivatives $\frac{\partial f}{\partial x_i}$ exist at P and are continuous in an open disk about P, where an open disk of radius r about P means the set of all points in $\mathbb{R}^n$ whose distance from P is less than r.

Friday, September 5. We began class with a discussion concerning how to extend the notion of differentiability to more general functions, including scalar and vector functions of several variables. After showing how the definition of differentiability for scalar functions of three and n variables mirrors the definition of differentiability for functions of two variables, we gave the ultimate definition was as follows:

General Definition of Differentiability. Suppose $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $P = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and write $\underline{x}$ for $(x_1, \ldots, x_n)$. Then $F(\underline{x})$ is differentiable at P if:

(i) Each partial derivative $\frac{\partial f_i}{\partial x_j}(P)$ exists and
(ii) $\lim_{\underline{x}\to P} \frac{||F(\underline{x}) - L(\underline{x})||}{||\underline{x} - P||} = 0$, where $L(\underline{x}) = (L_1(\underline{x}), \ldots, L_n(\underline{x}))$ and each

$$L_i(\underline{x}) = \frac{\partial f_i}{\partial x_1}(P)(x_1 - a_1) + \cdots + \frac{\partial f_i}{\partial x_n}(P)(x_n - a_n) + f_i(P).$$

When then addressed the following question: If a function can be differentiable, is there a derivative? The answer was yes, the derivative at a point is just a matrix of partial derivatives. Thus, if $f(x,y)$ is differentiable at (a,b), the derivative $Df(a,b)$ is the 1×2 matrix $(f_x(a,b) \;\; f_y(a,b))$. In the general case, maintaining the notation above, if $F(\underline{x})$ is differentiable at P,then its derivative at P is the $m \times n$ matrix

$$DF(P) = (\frac{\partial f_i}{\partial x_j}(P)), \text{ with } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

So, for example, if $F(x,y,z) = (f(x,y,z), g(x,y,z))$, then $DF(a,b,c) = \begin{pmatrix} f_x(a,b,c) & f_y(a,b,c) & f_z(a,b,c) \\ g_x(a,b,c) & g_y(a,b,c) & g_z(a,b,c) \end{pmatrix}$. The point is that the derivative $DF(P)$ encodes exactly the information needed to write the best linear approximation of the function $F(\underline{x})$ near the point P.

Using the formula for the derivative, we then considered $F(x,y,z) = (x^3y + 3z, e^{xyz})$ and found that $DF(3,1,2) = \begin{pmatrix} 27 & 27 & 3 \\ 2e^6 & 6e^6 & 3e^6 \end{pmatrix}$.

We then began a discussion of optimization of functions of several variables, by noting that in the case of $f(x,y)$, a relative maximum or minimum value should occur where the tangent plane is parallel to the xy-plane, i.e., has the form $z = c$, for some $c \in \mathbb{R}$. This lead to the definition:

Definition. A point (a,b) in the domain of $f(x,y)$ is a *critical point* if either: (i) $f_x(a,b) = 0 = f_y(a,b)$ or (ii) One of $f_x(a,b)$ or $f_y(a,b)$ is not defined.

We ended class by finding the critical point for the functions:

(i) $f(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$, which has critical point $(-1,-1)$.
(ii) $f(x,y) = \sqrt{xy}$, whose critical points are the x and y axes.

Monday, September 8. We began class by reviewing the definition of critical point for a function $f(x,y)$. These were points (a,b) in the domain of $f(x,y)$ for which $f_x(a,b) = 0 = f_y(a,b)$ or for which one of f_x or f_y are undefined at (a,b).

We then defined, what it means for a point to be a relative maximum or relative minimum of $f(x,y)$, or a saddle point on the graph of $f(x,y)$.

If there is an open disk D containing P such that $f(x_0, y_0) \geq f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative maximum** at P.

If there is an open disk D containing P such that $f(x_0, y_0) \leq f(x,y)$ for all points (x,y) that are in both D and S, then f has a **relative minimum** at P.

Let $P = (x_0, y_0)$ be in the domain of f where $f_x = 0$ and $f_y = 0$ at P. We say P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1, y_1) and (x_2, y_2) in D such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

Before moving on, we defined the second order partial derivatives of $f(x,y)$:

1. The **second partial derivative of** f **with respect to** x **then** x is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx}$$

2. The **second partial derivative of** f **with respect to** x **then** y is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$$

Similar definitions hold for $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

We noted that that in many cases, $f_{xy} = f_{yx}$, but this need not always hold. We stated, but did not discuss at length, that $f_{xy}(a, b) = f_{yx}(a, b)$ provided both mixed partial derivatives are continuous in an open disk about (a, b). This condition is implicit in the second derivative test below. As a class, we then calculate all second order partial derivatives of $f(x, y) = e^{x^2+3y^3}$.

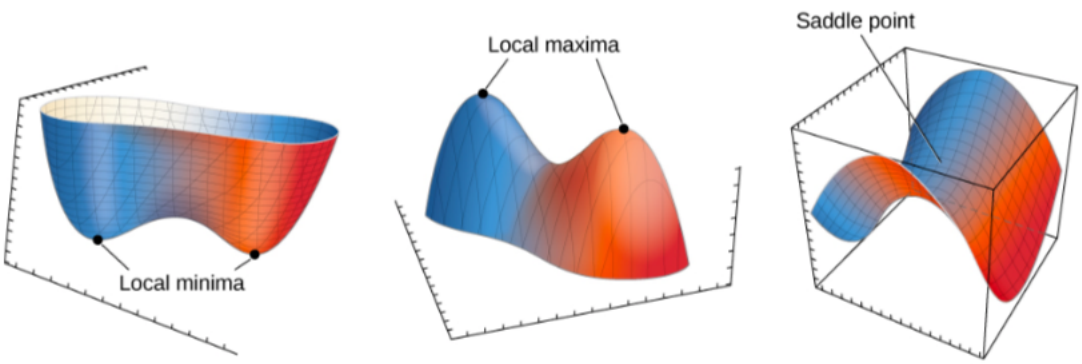
We then stated:

Theorem 12.8.2 Second Derivative Test

Let R be an open set on which a function $z = f(x, y)$ and all its first and second partial derivatives are defined, let $P = (x_0, y_0)$ be a critical point of f in R, and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at P.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at P.
3. If $D < 0$, then f has a saddle point at P.
4. If $D = 0$, the test is inconclusive.



Using the second derivative test, we then classified the critical points for $g(x, y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$ and found the critical points for $f(x, y) = (x^2 - y^2)e^{-\frac{1}{2}(x^2+y^2)}$ and stated that $(0, 0)$ is a saddle point, $f(\pm\sqrt{2}, 0)$ are relative maxima and $f(0, \pm\sqrt{2})$ are relative minima.

Wednesday, September 10. We began class by reviewing the statement of the second derivative test given in the previous lecture. We then considered the following problem: Find the dimensions of the rectangular box with minimum surface area, subject to the constraint that its volume is 100cm^3. We noted the one critical point for the surface area found was necessarily a minimum, since boxes with fixed volume can have arbitrarily large surface area. We verified this using the second derivative test.

We then illustrated the second derivative test by analyzing the the values of $f(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$ near the origin by completing a square, and considering the case when $\alpha\gamma - \beta^2$ is greater than zero, arriving at conclusions that mirrored the second derivative test.

We then began a discussion concerning absolute maxima and absolute minima for a function of two variables. We recalled, that for a function $f(x)$ of one variable, in order to guarantee that $f(x)$ has absolute extreme values, one must assume that $f(x)$ is continuous on a closed interval $[c, d]$. To find these values, one must find critical points on the interior of the interval $[c, d]$ and evaluate $f(x)$ at each of these points, and then one must calculate$f(c)$ and $f(d)$. The largest of these values is the absolute maximum of $f(x)$ on $[c, d]$ and smallest is the absolute minimum of $f(x)$ on $[c, d]$. We then explained that one needs similar condition for functions of two variables. Namely, the function in question must be continuous and the domain in $\mathbb{R}^2$ must be *closed* and *bounded*.

We next defined the conceptss of *absolute maximum* and *absolute minimum*: Given a subset $D \subseteq \mathbb{R}^2$, $f(x,y)$ has an *absolute maximum* (respectively, *absolute minimum*) at (a,b) if $f(x,y) \leq f(a,b)$ (respectively, $f(a,b) \leq f(x,y)$), for all $(x,y) \in X$. In this case, $f(a,b)$ is the absolute maximum (respectively, absolute minimum) value of $f(x,y)$ on X. We then gave the following definition:

Definition. Suppose that X is a subset of $\mathbb{R}^2$.

(i) X is *bounded* if there exists a closed disk $D \subseteq \mathbb{R}^2$ with $X \subseteq D$.
(ii) If X is bounded, then X is *closed* if it contained all of its boundary points.

Thus, for example,

(i) The open disk $X_1 = \{0 < x^2 + y^2 < 1\}$ is bounded, but not closed.
(ii) The infinite vertical strip $X_2 = \{(x,y) \mid 0 \leq x \leq 2\}$ is closed, but not bounded.
(iii) The rectangle $X_3 = \{(x,y) \mid -1 \leq x \leq 2,\ -1 \leq y \leq 1\}$ is both closed and bounded.

The important theorem concerning absolute extreme values is the following:

Theorem. *Let $X \subseteq \mathbb{R}^2$ be bounded and closed and $f(x,y)$ a continuous function defined on X. Then $f(x,y)$ has both an absolute maximum and absolute minimum value on X.*

We then explained, that the process for finding the absolute extreme values of $f(x,y)$ on X is similar to the one for functions of one variable: First find the critical points on the interior of X, and then find the absolute extreme values of $f(x,y)$ along the boundary of X. The largest and smallest of these values give the required absolute maximum and absolute minimum values of $f(x,y)$ on X. We explained that often, finding the absolute extreme values of $f(x,y)$ along the boundary of X reduces to the one variable case. We then illustrated the theorem for $f(x,y) = x^2 + y^2$ and D the unit closed disk in $\mathbb{R}^2$ and for $f(x,y) = x^2 + 2xy$ and D the rectangle in $\mathbb{R}^2$ with vertices (2, -1), (2, 2), (-1, 2), (-1, -1).

Friday, September 12. We began class with discussion of why second derivative test works, augmenting the discussion we had in the previous lecture, concerning quadratic functions and good quadratic approximations.

$$(*) \quad f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}\{f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\}.$$

If we let $Q(x,y)$ denote the right hand side of the expression above, i.e., $Q(x,y)$ is the good quadratic approximation of $f(x,y)$ near (a,b), Taylor's theorem states that if all first and second partials exists and are continuous in an open disk containing (a,b), then

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - Q(x,y)}{|(x,y) - (a,b)|^2} = 0,$$

which is stronger than the condition required for differentiability at (a,b). If we take h_1, h_2 sufficiently small, then (*) yields,

$$(**) \quad f(a+h_1, b+h_2) \approx f(a,b) + f_x(a,b)h_2 + f_y(a,b)h_2 + \frac{1}{2}\tilde{Q}(h_1,h_2),$$

where $\tilde{Q}(h_1,h_2) = f_{xx}(a,b)h_1^2 + 2f_{xy}(a,b)h_1h_2 + f_{yy}(a,b)h_2^2$. We then noted that our algebraic discussion from the previous lectures shows that conditions $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ hold, then $\tilde{Q}(h_1,h_2) > 0$ for all h_1, h_2 sufficiently small, showing that if $f_x(a,b) = 0 = f_y(a,b)$, then $f(a,b) < f(a+h_1, b+h_2)$ for all h_1, h_2 sufficiently small, implying that $f(x,y)$ has a local minimum at (a,b). Similarly, the other conditions in the second derivative test imply $\tilde{Q}(h_1,h_2) < 0$, for all sufficiently small h_1, h_2 or $\tilde{Q}(h_1,h_2)$ takes both positive and negative values, yielding a relative maximum or saddle point at (a,b).

We then discussed finding extreme values for functions $f(x,y,z)$ of three variables. We noted that a point $P = (a,b,c)$ is a critical point if $f(x,y,z)$ if it is either a solution to the system of equations

$$f_x(x,y,z) = 0$$
$$f_y(x,y,z) = 0$$
$$f_z(x,y,z) = 0,$$

or one of the first order partials is undefined at P. We then defined $D(P) = f_{xx}(P)f_{yy}(P) - f_{xy}(P)^2$ and

$$H(P) = \begin{vmatrix} f_{xx}(P) & f_{xy}(P) & f_{xz}(P) \\ f_{yx}(P) & f_{yy}(P) & f_{yz}(P) \\ f_{zx}(P) & f_{zy}(P) & f_{zz}(P) \end{vmatrix}.$$

$H(P)$ is called the *Hessian* of $f(x, y, z)$ at P.

Second Derivative Test. Suppose $P = (a, b, c)$ critical point that is a solution to the system of equations above and all second order partial derivatives of $f(x, y, z)$ are continuous in a disk about P. Then:

(i) If $f_{xx}(P), D(P), H(P)$ are all greater than zero, $f(x, y, z)$ has a relative minimum at P.
(ii) If $f_{xx}(P) < 0, D(P) > 0, H(P) < 0$, then $f(x, y, z)$ has a relative maximum at P.
(iii) If $H(P) \neq 0$, and neither (i) nor (ii) holds, then $f(x, y, z)$ has a saddlle point at P.
(iv) If $H(P) = 0$, the test is inconclusive.

We then began a discussion of the chain rule for multivariable functions, starting with the simplest version where $f(x, y) = f(x(t), y(t))$, leading to the formula

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

We illustrated this formula by calculating $\frac{df}{dt}$, for $f(x, y) = 3x^2y^3$, with $x = 2t + 1$ and $y = 2t^2$. We then noted the following more general form: If $f = f(x_1, \ldots, x_n)$ and each $x_i = x_i(t)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = f_{x_1} \cdot x_1'(t) + \cdots + f_{x_n} \cdot x_n'(t).$$

This ultimately lead to the most general form:

General for of the chain rule. Suppose $f = f(x_1, \ldots, x_n)$ and each $x_i = x_i(u_1, \ldots, u_m)$, then

$$\frac{\partial f}{\partial u_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_j}{\partial u_j},$$

for all $j = 1, 2, \ldots, m$. We then had the class calculate (in groups) f_u, f_v, f_w for $f(x, y, z) = 3x^3y^3z$ with $x = uv + w^2, y = \cos(2u + wv), z = u^3 + v^3 + w^3$ (or something similar).

Monday, September 15. We began class by defining the directional derivative of the function $f(x, y)$ at (a, b) in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$:

$$D_{\vec{u}}f(a, b) = \lim_{h\to 0} \frac{f(a + hu_1, b + u_2h) - f(a, b)}{h},$$

noting that this is the rate of change of $f(x, y)$ at (a, b) along the line $(a, b) + t \cdot \vec{u}$, assuming the limit exists. We emphasized the importance of taking a unit vector in this definition, so that the quantity calculated only depends upon the function f and the direction of the direction vector, and not also on the magnitude of the direction vector.

We then used this definition to calculate $D_{\vec{u}}f(a, b)$ for $f(x, y) = x^2y$ in the direction of $\vec{u} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$,

$$\begin{aligned} D_{\vec{u}}f(a, b) &= \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \\ &= \lim_{h\to 0} \frac{(a + h\frac{\sqrt{2}}{2})^2(b + h\frac{\sqrt{2}}{2}) - a^2b}{h} \\ &= \lim_{h\to 0} \frac{a^2b + abh\sqrt{2} + h^2 \cdot \frac{b}{2} + a^2h\frac{\sqrt{2}}{2} + ah^2 + \frac{\sqrt{2}}{4}h^3 - a^2b}{h} \\ &= \lim_{h\to 0} ab\sqrt{2} + h \cdot \frac{b}{2} + a^2\frac{\sqrt{2}}{2} + ah + \frac{\sqrt{2}}{4}h^2 \\ &= \sqrt{2}ab + \frac{\sqrt{2}}{2}a^2. \end{aligned}$$

We observed that this last expression is $(2ab\vec{i} + b^2\vec{j}) \cdot \vec{u} = (\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}) \cdot \vec{u}$. We then noted that in general, one can calculate the directional derivative as

$$D_{\vec{u}}f(a,b) = (\frac{\partial f}{\partial x}(a,b)\vec{i} + \frac{\partial f}{\partial y}(a,b)\vec{j}) \cdot \vec{u}.$$

This lead to the definition of the *gradient*, ∇f, of a scalar function f:

(i) For $f(x,y)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$.
(iii) For $f(x,y,z)$, $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.
(iii) For $f(x_1, x_2, \ldots, x_n)$, $\nabla f = \frac{\partial f}{\partial x_1}\vec{e_1} + \frac{\partial f}{\partial x_2}\vec{e_2} + \cdots + \frac{\partial f}{\partial x_n}\vec{e_n}$, where the vector $\vec{e_i}$ is the vector in $\mathbb{R}^n$ all of whose coordinates are zero, except the ith coordinate, which is 1.

Using this notation, then in the cases above we have

(i) $D_{\vec{u}}f(a,b) = \nabla f(a,b) \cdot \vec{u}$.
(ii) $Duf(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$.
(iii) $D_{\vec{u}}f(a_1, a_2, \ldots, a_n) = \nabla f(a_1, a_2, \ldots, a_n) \cdot \vec{u}$.

where in each case $\vec{u}$ is an appropriate unit vector.

We then mentioned that ∇ is a *differential operator* that turns scalar value functions into vector valued function through the differentiation process. As such, one can expect ∇ to have similar properties that hold upon differentiation. Indeed, the following properties hold:

(i) $\nabla(f+g) = \nabla f + \nabla g$.
(ii) $\nabla(cf) = c\nabla f$, for the constant c.
(iii) $\nabla(fg) = f\nabla g + g\nabla f$.
(iv) For $h(t)$, $\nabla h(f) = h'(f)\nabla f$.

We ended class with the following **important fact**

For the function $f(x,y,z)$, $D_{\vec{u}}f(a,b,c)$ achieves its greatest value when $\vec{u}$ points in the same direction as $\nabla f(a,b,c)$, and moreover, the rate of change in that direction is $||\nabla f(a,b,c)||$. Likewise, we noted that $-\nabla f(a,b,c)$ points in the direction in which the rate of change at (a,b,c) is the least, and that rate of change is $-||\nabla f(a,b,c)||$. This followed easily from $\nabla f(a,b,c) \cdot \vec{u} = ||\nabla f(a,b,c)|| \cdot ||\vec{u}|| \cos(\theta) = ||\nabla f(a,b,c)|| \cos(\theta)$, where θ is the angle between $\nabla f(a,b,c)$ and $\vec{u}$.

Wednesday, September 17. We began class by making a few announcement relevant to the upcoming midterm exam. We then continued by showing that if $f(x,y,z) =$ **Constant** is a level surface, then $\nabla f(a,b,c)$ is normal to the surface at the point $P = (a,b,c)$. Thus, the plane tangent to the surface at P is given by the equation

$$0 = \nabla f(P) \cdot \{(x-a)\vec{i} + (y-b)\vec{j} + (z-c)\vec{k}\} = f_x(P) \cdot (x-a) + f_y(P) \cdot (y-b) + f_z(P) \cdot (z-c).$$

The rest of the cclass was devoted to a discussion of the following theorem.

Clairaut's Theorem. Suppose $f(x,y)$ has continuous first and second order partials in an open disk D around the point (a,b) in its domain. Then $f_{xy}(a,b) = f_{yx}(a,b)$. In fact, for all $(x_0, y_0) \in D$, we have $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

We first illustrated this theorem with $f(x,y) = e^{x^2+3y^2}$, noting that one has equality of mixed partials at all points in $\mathbb{R}^2$. We then showed that $f_{xy}(0,0)$ and $f_{yx}(0,0)$ exist, but are *not* equal, for the function

$$f(x,y) = \begin{cases} x^2\arctan(\frac{y}{x}) - y^2\arctan(\frac{x}{y}), & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}.$$

We observed that, in general, a potential problem stems from the fact that for a function $L(h,k)$, the two iterated limits $\lim_{k\to a}\lim_{h\to b} L(h,k)$ and $\lim_{h\to b}\lim_{k\to a} L(h,k)$ need not be equal - for example, the iterated limits with both h,k approaching 0 are not equal for $L(h,k) = \frac{h+k}{h-k}$.

Friday, September 19. The class worked in groups on practice problems for the first midterm exam.

Monday, September 22. The class worked in groups on practice problems for the first midterm exam.

Wednesday, September 24. Jake began a discussion of the method Lagrange multipliers, which is a technique for finding maximum or minimum values of a function of several variables subject to a constraint. For example, one seeks to find the maximum or minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$, for a constant c. One sets $\nabla f = \lambda \nabla g$. This together with the constraint equation gives four equations in the unknowns x, y, z, λ. Solutions involving x, y, z give rise to critical points which are then tested in $f(x, y, z)$ to find the required minimum and maximum values.

Friday, September 26. We began class with a discussion concerning the results of Exam 1 and some comments related to particular problems on the exam. We then continued with the discussion of the method of Lagrange multipliers. We first showed how Lagrange multipliers give a much easier solution to the following problem, appearing on a worksheet a couple of weeks back: Show that for boxes with fixed surface area, the volume is a maximum when the box is a cube. We then discussed the optimization of $f(x, y, z)$ subject to two constraints, $g_1(x, y, z) = c_1$ and $g_2(x, y, z) = c_2$. The goal in this case is to find solutions to the system of equations coming from $\nabla f = \lambda \nabla g + \lambda_2 \nabla g_2$ together with the constraint equations. We then gave an example of this by finding the points on the curve C nearest and farthest from (0,0,0), where C is the curve obtained by intersecting the cone $z^2 = x^2 + y^2$ with the plane $z = x + y - 2$. One key observation was that rather than finding critical points on the curve using the object function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ it suffices to find critical points on the curve using the object function $\tilde{f}(x, y, z) = x^2 + y^2 + z^2$.

Monday, September 29. Today we started our discussion of multiple integration. We began class by reviewing the situation for definite integrals of the form $\int_a^b f(x)\ dx$. We first recalled the definition of the definite integral as a limit of partial sums taken over successively finer partitions of the interval $[a, b]$. We then demonstrated how the Fundamental Theorem of Calculus works, i.e., we sketched a proof of the crucial formula $\int_a^b f(x)\ dx = F(b) - F(a)$, for $F(x)$ satisfying $F'(x) = f(x)$. The point of this discussion being two fold: It is important to understand conceptually what a definite integral is, while on the other hand, one needs to know how to calculate a definite integral.

We then showed that one can define the definite integral of $f(x, y)$ over a closed rectangle R as a limit of partial sums of a similar type, only now one takes finer rectangular partitions of the domain R, obtaining a limit of sums of the form $\Sigma_i \Sigma_j f(c_i, d_j) \Delta x \Delta y$. This limit, if it exists, is denoted $\int\int_R f(x, y)\ dA$. We also described how the double integral is defined over a more general region R, by covering R with small rectangles, or any shaped objects R_i with small area. One then chooses $p_i \in R_i$ and takes a limit of partial sums $\sum_i f(P_i) \cdot \text{area}(R_i)$. We ended class by noting Fubini's theorem:

Fubini's Theorem for rectangles. Suppose $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$. Then

$$\int\int_R f(x, y)\ dA = \int_c^d \{\int_a^b f(x, y)\ dx\}\ dy = \int_a^b \{\int_c^d f(x, y)\ dy\}\ dx,$$

where $\int_c^d \{\int_a^b f(x, y)\ dx\}\ dy$ and $\int_a^b \{\int_c^d f(x, y)\ dy\}\ dx$ are *iterated integrals*.

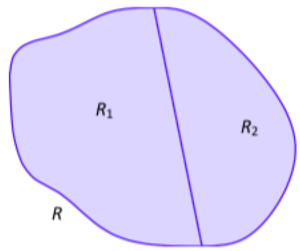
We discussed how to calculate iterated integrals by integrating one variable at a time, keeping the second variable fixed during the first integration process. We closed with an application of this to the integral $\int\int_A f(x, y)\ dA$, where R is the rectangle $0 \le x \le 1, 1 \le y \le 4$:

Wednesday, October 1. We began class by reviewing Fubini's theorem for integrating $f(x, y)$ over a rectangle and calculated $\int\int_R xye^{x^2+y^2}\ dA$ in two ways, for $R = [0, 1] \times [2.3]$. We then presented the following properties

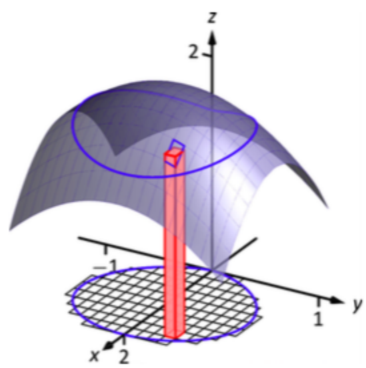
Properties. Assume $f(x, y)$, $g(x, y)$ are integrable over the rrectangular region R. Then:

(i) $\int\int_R \{f(x, y) \pm g(x, y)\}\ dA = \int\int_R f(x, y)\ dA \pm \int\int_R g(x, y)\ dA$.
(ii) $\int\int_R \lambda \cdot f(x, y)\ dA = \lambda \cdot \int\int_D f(x, y)\ dA$, for all $\lambda \in \mathbb{R}$.

(iii) $\int\int_R = \int\int_{R_1} f(x,y)\ dA + \int\int_{R_2} f(x,y)\ dA$, where $R = R_1 \cup R_2$ and either R_1, R_2 are disjoint, or only intersect along their boundaries

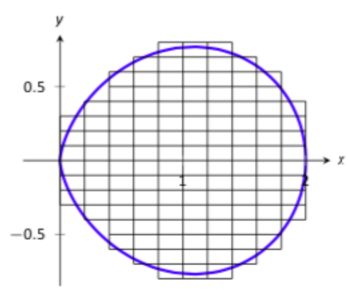


(iv) If $f(x,y) \geq 0$, for all $(x,y) \in R$, then $\int\int_D f(x,y)\ dA$ represents the volume of the region in $\mathbb{R}^3$ bounded above by the graph of $f(x,y)$ and bounded below by R. The point being that a partial sum taken over a rectangular partition of R reprsents a sum of vlumes of cubes of the type below. Taking the limit of such sums gives the indicated volume.



(b)

We then had a brief a discussion of what $\int\int_D f(x,y)\ dA$ should mean, where $R \subseteq \mathbb{R}^2$ is a possible domain of integration. Proceeding by analogy, we observed that the notation is suggestive: we should be summing (via a double sum) values of $f(x,y)$ times small increments of area. For this we described the process of covering the region R with small rectangles $\Delta x_i \times \Delta y_j$, something like this:



We selected a point (c_i, d_j) from each $\Delta x_1 \times \Delta y_j$ rectangle and formed the Riemann sum $\sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j$. We then defined

$$\int\int_D f(x,y)\ dA = \lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j f(c_i, d_j)\Delta x_i \Delta y_j,$$

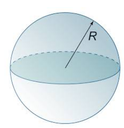
provided the limit exists. The resulting number is then *double integral of f(x,y) over the region R.*

Next we stated the following version of Fubini's theorem that enables us to calculated double integrals over more general regions:

> **Theorem 13.2.2 Fubini's Theorem**
>
> Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.
>
> 1. If R is bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then
>
> $$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$
>
> 2. If R is bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then
>
> $$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

We ended class by calculating $\int\int_R x^2y + e^x \; dA$ in two ways, for R the triangle with vertices (0,0), (1, 0), (1,1). We also set up the integral $\int\int_R e^{x^2} \; dA$ in two ways, noting that by integrating with respect to y first, the integral was doable, while the integral was not doable integrating with respect to x first.

Friday, October 3. We began class by cinsidering the question of calculating the volume of the sphere of radius R centered at the origin. (In class, we used ρ for the radius.)



We can integrate the function $f(x, y) = \sqrt{R^2 - x^2 - y^2}$ over the closed disk $D : 0 \le x^2 + y^2 \le R^2$. This will give us the volume of the top half of our sphere. We can think of D as a region of Type 2, being bounded above by the curve $y = \sqrt{R^2 - x^2}$ and bounded below by the curve $y = -\sqrt{R^2 - x^2}$, with $-R \le x \le R$.

Thus, the volume of the sphere is given by

$$2\int\int_D \sqrt{R^2 - x^2 - y^2} \; dA = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2} \; dy \; dx.$$

Note that the first of the two iterated integrals requires consideration of an indefinite integral of the form $\int \sqrt{a^2 - y^2} \; dy$, where $a = \sqrt{R^2 - x^2}$.

This can be worked out using a trig substitution like $y = a\sin(u)$, and the answer becomes

$$\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}(\frac{y}{2}).$$

We must then replace a by $\sqrt{R^2 - x^2}$, take the difference of y evaluated at $\sqrt{R^2 - x^2}$ and $-\sqrt{R^2 - x^2}$, and then integrate with respect to x.

There is a better solution!

The idea is a two variable form of u-substitution, namely, we use polar coordinates as follows: Set $x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = rdrd\theta$. We will explain this latter equality, in a future lecture, but the point is that just like in u-substitution, we don't simply exchange dx for du, here we do not simply exchange dA for $drd\theta$, as there is a scaling factor of r involved. In terms of r and θ, D is described as: $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Upon substituting, we get:

$$\begin{aligned}
2\int\int_D \sqrt{R^2-x^2-y^2}\ dA &= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-(r\cos(\theta))^2-(r\sin(\theta))^2}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R \sqrt{R^2-r^2(\cos^2(\theta)+\sin^2(\theta))}\ r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\ dr\ d\theta
\end{aligned}$$

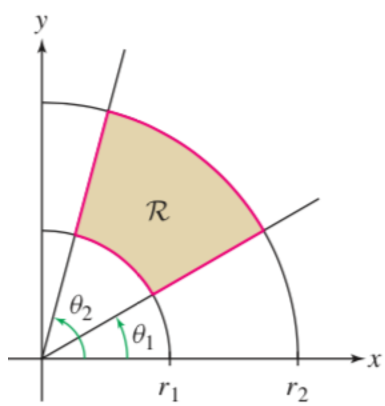
Note that now the domain of integration is a rectangle in the (r,θ) plane. An easy u-substitution shows that $\int r\sqrt{R^2-r^2}\ dr = -\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}$. Thus:

$$\begin{aligned}
2\int_0^{2\pi}\int_0^R r\sqrt{R^2-r^2}\ dr\ d\theta &= 2\int_0^{2\pi} -\frac{1}{3}(R^2-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=R}\ d\theta \\
&= 2\int_0^{2\pi} 0+\frac{R^3}{3}\ d\theta \\
&= 2\cdot\frac{R^3}{3}\theta\Big|_{\theta=0}^{\theta=2\pi} \\
&= 2\cdot\frac{2\pi R^3}{3} \\
&= \frac{4\pi}{3}R^3.
\end{aligned}$$

We then considered the following example and had the class try to set up the corresponding integral using polar coordinates with $x=r\cos\theta, y=r\sin\theta, dA=r\ dr\ d\theta$.

Example 1. Find the volume below the surface given by $z=4-(x^2+y^2)$ and above the xy-plane.

We then had a discussion explaining why dA is replaced by $rdrd\theta$ when using polar coordinates. Working in polar coordinates, it makes sense to subdivide the domain of integration D into small *polar rectangles* with $\theta_1\le\theta\le\theta_2$ and $r_2\le r\le r_2$, D is covered by regions $\mathcal{R}$ that look like:



When we form a Riemann sum, we must multiply a function value on $\mathcal{R}$ by the area of $\mathcal{R}$. The area of $\mathcal{R}$ is:

$$\frac{r_2^2}{2}\cdot(\theta_2-\theta_1)-\frac{r_1^2}{2}\cdot(\theta_2-\theta_1).$$

Now set $\theta_2-\theta_1=\Delta\theta$, $r=r_1$ and $r_2=r+\Delta r$. Then:

$$\begin{aligned}
\text{area}(\mathcal{R}) &= \frac{(r+\Delta r)^2}{2}\cdot\Delta\theta-\frac{r^2}{2}\cdot\Delta\theta \\
&= r\ \Delta r\ \Delta\theta+\frac{(\Delta r)^2\Delta\theta}{2}.
\end{aligned}$$

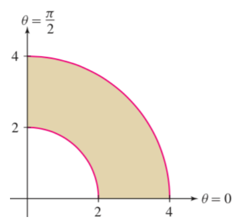
When Δr and $\Delta\theta$ are small, the term $\frac{(\Delta r)^2 \Delta\theta}{2}$ is much smaller than the term $r\ \Delta r\ \Delta\theta$. Thus:

$$\text{area}(\mathcal{R}) \approx r\ \Delta r\ \Delta\theta.$$

This approximation gets better as Δr and $\Delta\theta$ tend to zero. Thus dA, measured in polar coordinates, becomes $r\ dr\ d\theta$. We can use these approximations in the Riemann sums defining the double integral, which in the limit becomes an iterated integral $\int\int f(r\cos(\theta), r\sin(\theta))\ r\ dr\ d\theta$.

We then set of the following example.

Example 2. Calculate $\int\int_R x + y\ dA$, where D is the region



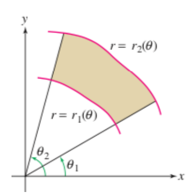
.

We noted that, in polar coordinates, D can be described as $0 \leq \theta \leq \frac{\pi}{2}$ and $2 \leq r \leq 4$. Thus:

$$\begin{aligned}\int\int_D x + y\ dA &= \int_0^{\frac{\pi}{2}} \int_2^4 (r\cos(\theta) + r\sin(\theta)) r\ dr\ d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_2^4 r^2 (cos(\theta) + \sin(\theta))\ dr\ d\theta\end{aligned}$$

which is easily calculated.

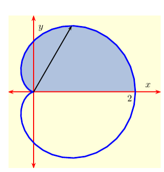
Fubini's Theorem over more general polar regions. Suppose we wish to integrate the continuous function $f(x, y)$ over a region D of the following type:



Here D is given by $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_1 \leq \theta \leq \theta_2$, where each $r_i(\theta)$ is a function of θ. Then we have

$$\int\int_D f(x, y)\ dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos(\theta), r\sin(\theta))\ r dr d\theta.$$

Example 2. Calculate $\int\int_D y\ dA$, where D is the set of points lying above the x-axis and inside the cardioid $r = 1 + \cos(\theta)$:



Solution:

$$\int\int_D y\ dA = \int_0^{\pi}\int_0^{1+\cos(\theta)} r\sin(\theta) r dr d\theta$$
$$= \int_0^{\pi}\int_0^{1+\cos(\theta)} r^2\sin(\theta)\ dr d\theta$$
$$= \int_0^{\pi} \frac{r^3}{3}\Big|_0^{1+\cos(\theta)} \sin(\theta)\ d\theta$$
$$= \frac{1}{3}\int_0^{\pi}(1+\cos(\theta))^3\sin(\theta)\ d\theta$$

The u-substitution $u = 1+\cos(\theta)$ yields

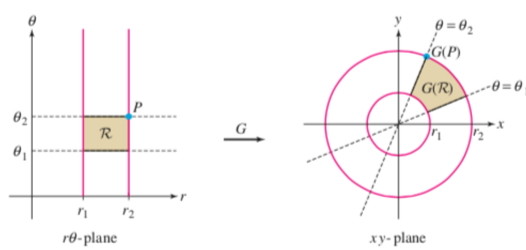
$$\int(1+\cos(\theta))^3\sin(\theta)\ d\theta = \int -u^3\ du = -\frac{1}{4}u^4 = -\frac{1}{4}(1+\cos(\theta))^4.$$

Thus:

$$\int\int_D y\ dA = \frac{1}{3}\{-\frac{1}{4}(1+\cos(\theta))^4\Big|_{\theta=0}^{\theta=\pi}\}$$
$$= -\frac{1}{12}\cdot\{(1+-1)^4-(1+1)^4\}$$
$$= \frac{16}{12}$$
$$= \frac{4}{3}.$$

We ended class by showing how to set up the double integral giving the volume of the region in $\mathbb{R}^3$ bounded above by the graphs of $z_1 = 8-(x^2+y^2)$ and $z_2 = x^2+y^2$.

Monday, October 6. We began our discussion of the change of variables principle for double integrals. We noted that one of the purposes of this principle is that it transforms a double integral over a domain of integration that may be difficult to integrate over into a double integral over a domain of integration that is more manageable. The example of this we have already seen is the use of polar coordinates. We can think of using polar coordinates as changing variables from x and y to r and θ. If we write $G(r,\theta) = (r\cos(\theta), r\sin(\theta))$, then we can think of G as a function that transforms verticle lines in the (r,θ)-plane to arcs on circles in the xy-plane .

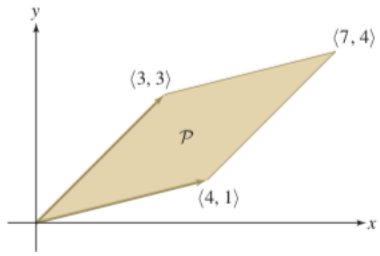


Note that $G(r,\theta)$ takes any vertical line $r = r_0$ in the r,θ-plane and wraps it infinitely many times around the circle of radius r_0 centered at the origin in the xy-plane. If $r = r_0$ and $0 \le \theta < 2\pi$, then G applied to this vertical line seqment in the r,θ-plane is the circle of radius r_0 (no points repeated) centered at (0,0) in the xy-plane. G also takes the rectangle $\mathcal{R}$ in the uv-plane, in the diagram above, to the polar rectangle $G(\mathcal{R})$ in the xy plane.

The r in the equation $dA = r\ dr\ d\theta$ comes from the *Jacobian* of the polar transformation

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{pmatrix}\cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta)\end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Example 1. Consider $\int\int_{\mathcal{P}} 3x + 2y \; dA$ for $\mathcal{P}$ the region:



A close look at $\mathcal{P}$ shows that if we try to think of $\mathcal{P}$ as a region of Type 1 or Type 2, we will have to subdivide $\mathcal{P}$ into three parts.

However, we can change variables.

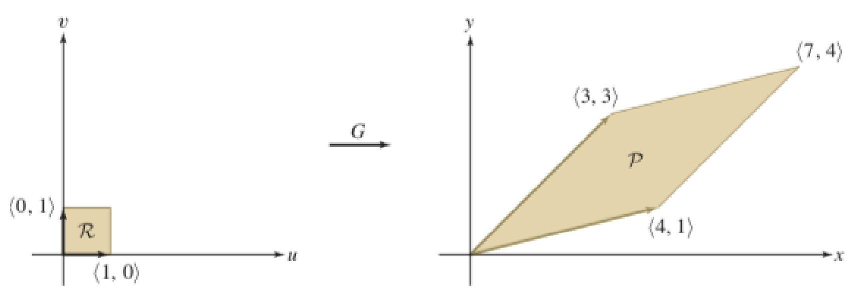
Set $x = 4u + 3v, y = u + 3v$, or equivalently, define $G(u, v) = (4u + 3v, u + 3v)$. We take the absolute value of the determinant of the 2×2 matrix of partial derivatives:

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix} = 9 = |9|$$

and set $dA = 9 \; du \; dv$. Now we substitute:

$$\begin{aligned} \int\int_{\mathcal{P}} 3x + 2y \; dA &= \int_0^1 \int_0^1 3(4u + 3v) + 2(u + 3v) \; 9dudv \\ &= 9 \int_0^1 \int_0^1 14u + 15v \; dudv \\ &= 9 \int_0^1 (7u^2 + 15uv) \Big|_{u=0}^{u=1} dv \\ &= 9 \int_0^1 7 + 15v \; dv \\ &= 9(7v + \frac{15}{2} v^2) \Big|_0^1 \\ &= 9(7 + \frac{15}{2}) \\ &= \frac{261}{2}. \end{aligned}$$

Where does this come from? We have a transformation (function) $G : \mathbb{R}^2 \to \mathbb{R}^2$ which takes (u, v) in the uv-plane to $(4u + 3v, u + 3v)$ in the xy-plane:



Let's see how G transforms $\mathcal{R}$ to $\mathcal{P}$. Note that $\mathcal{P}$ is the parallelogram spanned by the vectors (4,1) and (3,3). $G(0, 0) = (0, 0)$, $G(0, 1) = (3, 3)$,G(1,0) = (4,1), and $G(1, 1) = (7, 4)$ showing that G takes the corners of the unit square in the uv-plane to the corners of the parallelogram $\mathcal{P}$ in the xy-plane. .

We can also verify that G takes any point on the u-axis in the uv-plane to a point on the line through (0,0) and $(4,1)$. For example, $G(a,0)=(4a,a)$, which lies on the line $y=\frac{1}{4}x$. Note that if $0\le a\le 1$, then $(4a,a)$ lies on the line segment through (0,0) and (4,1). Thus, G transforms the lower edge of $\mathcal{R}$ to the line segment in $\mathcal{P}$ connecting $(0,0)$ and $(4,1)$.

Since $G(0,1)=(3,3)$ and $G(1,1)=(7,4)$, in a similar way one can see that G transforms each of the edges of $\mathcal{R}$ into corresponding edges of $\mathcal{P}$.

Finally, if (a,b) is in the interior of $\mathcal{R}$, then $0<a<1$ and $0<b<1$. The slope of the line through (0,0) in the xy-plane and $G(a,b)$ is $\frac{1}{4}\le\frac{u+3v}{4u+3v}\le 1$, which shows that $G(a,b)$ lies in the interior of $\mathcal{P}$.

Thus, G transforms $\mathcal{R}$ into $\mathcal{P}$.

Definition. (i) A transformation is a function $G(u,v)=(x(u,v),y(u,v))$, from the uv-plane to the xy-plane, The *Jacobian* of $G(u,v)$ is the function

$$\mathrm{Jac}(G):=\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}.$$

We will also write $\mathrm{Jac}(G)=\frac{\partial(x,y)}{\partial(u,v)}$. We will assume that our transformations satisfy the property that *all first order partial derivatives* $\frac{\partial x}{\partial u},\frac{\partial x}{\partial v},\frac{\partial y}{\partial u},\frac{\partial y}{\partial v}$ *exist and are continuous* in the domain of $G(u,v)$.

(ii) The transformation $G(u,v)$ is said to be *one-to-one* if no two points in the uv-plane go to the same point in the xy-plane under the transformation $G(u,v)$. i,e., $G(u_1,v_1)=G(u_2,v_2)$ implies $(u_1,v_1)=(u_2,v_2)$.

Here is the theorem that tells us how to change variables in a double integral.

Change of Variables Theorem. Let $G(u,v)=(x(u,v),y(u,v))$ be a transformation from the uv-plane to the xy-plane. Suppose R_0 is a subset of the uv-plane and write $R=G(R_0)$. Assume $G(u,v)$ is one-to-one on the interior of R_0. Then:

$$\begin{aligned}\int\int_R f(x,y)\;dA &= \int\int_{R_0} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\;dA\\ &= \int\int_{R_0} f(x(u,v),y(u,v))\;|Jac(G)|\;dudv\end{aligned}$$

where $|\mathrm{Jac}(G)|=|\frac{\partial(x,y)}{\partial(u,v)}|$ denotes the absolute value of the Jacobian of G. The crucial point in this formula is that small portions of area dA in the xy-plane become $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ times small portions of area dA in the uv-plane. Note by definition, the first order partials of $G(u,v)$ are assumed to be continuous.

Wednesday, October 8. We continued our discussion of the change of variables theorem for double integration. After recalling the translation transformation, we discussed at length linear transformations:

Linear Transformations. A transformation T from the uv-plane to the xy-plane is said to be a *linear transformation* if $T(u,v)=(au+bv,cu+dv)$, for constants $a,b,c,d\in\mathbb{R}$. Note that in this case we have

$$\mathrm{Jac}(T)=\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-bc.$$

Important Properties of linear transformations.

(i) $T(P_1+P_2)=T(P_1)+T(P_2)$, for points P_1,P_2 in the uv-plane.
(ii) $T(\lambda P)=\lambda T(P)$, for all points P in the uv-plane and $\lambda\in\mathbb{R}$.
(iii) T is one-to-one if and only if the Jacobian $ad-bc\neq 0$

Part (ii) shows that T takes lines through the origin in the uv-plane to lines through the origin in the xy-plane. In general, $T=(au+bv,cu+vd))$ transforms the unit square in the uv-plane to the parallelogram in the xy-plane spanned by the vectors (a,c) and (b,d). We noted that this can be seen using the same reasoning as in Example 1 from the previous lecture.

We also considered a second type of transformation,

Translation. Let $G(u,v) = (u+a, v+b)$. Then this is the transformation obtained by translating the origin of the uv-plane to the point (a,b) in the xy-plane. We noted that such a transformation should not change area

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

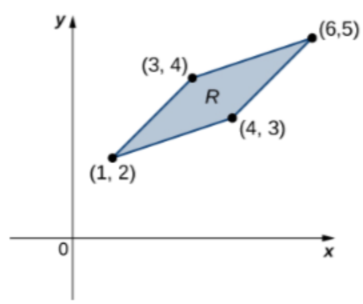
For example: If $u^2+v^2 = 2^2$, and $G(u,v) = (u+a, v+b)$, then $u = x-a$ and $v = y-b$, so

$$(x-a)^2 + (y-b)^2 = 2^2.$$

In other words, G translates the circle (and the disk) of radius 2 in the uv-plane centered at (0,0) to the circle (and disk) of radius 2 in the xy-plane, centered at (a,b). We illustrated this by calculating $\int\int_R \sqrt{(x-2)^2+(y-3)^2}\, dA$ where R is the disk of radius two centered at (2,3). By setting $G(u,v) = (u+2, v+3)$, the given integral becomes $\int\int_{R_0} \sqrt{u^2+v^2}\, dA$, where R_0 is the disk of radius two centered at the origin in the uv-plane. This latter integral can be solved using polar coordinates, which itself is another transformation. Alternately, one can do one transformation by setting $G = (u\cos(v)+2, u\sin(v)+3)$. We also used similar ideas to calculate the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Here is an example combining both a linear transformation with a translation.

Example 1. Determine the transformation which takes the unit square in the uv-plane to the parallelogram



in the xy-plane.

Solution: We see that this parallelogram is the translation of one similar to it with lower left corner at the origin. If we move the vertex (1,2) to the origin,we get a new parallelogram with vertices (0,0), (2,2), (5,3), (3,1), moving counterclockwise along the perimeter. This new parallelogram is spanned by the vectors (3,1) and (2,2), so that by the previous lecture, $T(u,v) = (3u+2v, u+2v)$ takes the unit square in the uv-plane to the new parallelogram. If we add (1,2) to the new coordinates, we get

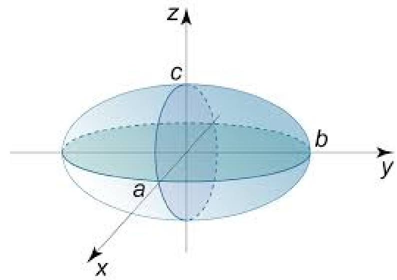
$$G(u,v) = (3u+2v+1, u+2v+2),$$

and this transformation takes the unit square in the uv-plane to the original parallelogram in the xy-plane.

Notice that G takes the vertices (0,0), (1,0), (1,1), (0,1), in the uv-plane to the vertices (1,2), (4,3), (6,5), (3,4) in the xy-plane. Moreover, $\text{Jac}(G) = \det\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = 4$.

We ended class with a discussion of the transformations needed to calculate the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Example 2. Calculate the volume of the ellipsoid E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



Solution: If we let D be the elliptic disk $0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then:

$$\text{vol}(E) = 2 \int\int_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dA.$$

We can do this two different ways using the change of variables formula. First, we can use the transformation from the end of last lecture, $G(u,v) = (au\cos(v), bu\sin(v))$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = abu$, which takes the rectangle $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$ to D. We then get:

$$\begin{aligned}
\text{vol}(E) &= 2\int_0^{2\pi}\int_0^1 c\sqrt{1 - \frac{(au\cos(\theta))^2}{a^2} - \frac{(bu\sin(\theta))^2}{b^2}}\ abu\ dudv \\
&= 2abc\int_0^{2\pi}\int_0^1 u\sqrt{1-u^2}\ dudv \\
&= 2\int_0^{2\pi} -\frac{1}{3}(1-u^2)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ dv \\
&= 2abc\int_0^{2\pi}\frac{1}{3}\ dv \\
&= \frac{4}{3}\pi abc
\end{aligned}$$

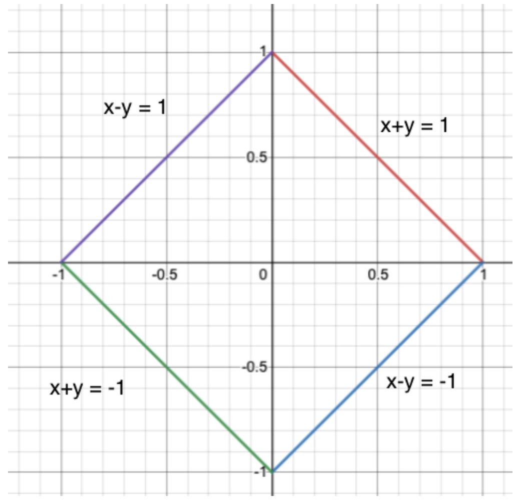
Alternately, we can use the linear transformation $T(u,v) = (au, bv)$, with $|\frac{\partial(x,y)}{\partial(u,v)}| = ab$. This transformation stretches the plane a units horizontally and b units vertically. It takes the unit disk $D' : 0 \leq u^2 + v^2 \leq 1$ to D. Thus:

$$\begin{aligned}
\text{vol}(E) &= 2\int\int_D c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dA. \\
&= 2c\int\int_{D'}\sqrt{1 - \frac{(au)^2}{a^2} - \frac{(bv)^2}{b^2}}ab\ dudv \\
&= 2abc\int\int_{D'}\sqrt{1-u^2-v^2}\ dudv.
\end{aligned}$$

Now, we can either use polar coordinates to evaluate the last double integral, or in this case, recognize it as the volume of the top half of the unit sphere, which is $\frac{2}{3}\pi$. Thus, $\text{vol}(E) = \frac{4}{3}\pi abc$.

Friday, October 10. We began with an example which shows that a change of variables can be used when the integrand has no (obvious) antiderivative.

Example 1. Calculate $\int\int_D (x+y)^2 e^{x^2-y^2}\ dA$, where D is the diamond with vertices (1,0), (-1,0), (0,1), (0,-1).



Solution. Note that as written, the integrand does not have an antiderivative with respect to either variable. If we could find a transformation $G(u, v)$ having the property that $x + y = u$ and $x - y = v$, then the integrand becomes $u^2 e^{uv}$, which is manageable. This suggests that to find $G(u, v)$, we must solve for x and y in terms of u and v. Adding the two equations gives $2x = u + v$, so $x = \frac{u+v}{2}$. Subtracting gives $2y = u - v$, so $y = \frac{u-v}{2}$. Therefore, we take $G(u, v) = (\frac{u+v)}{2}, \frac{u-v}{2})$ is linear so the pre-image D_0 of D must at least be a parallelogram. The corners of D are (1,0), (-1,0), (0,1), (0,-1). Substituting these points into the equations for u and v gives, (1,1), (-1,-1), (1,-1), (-1,1). Thus, D_0 is the rectangle $[-1, 1] \times [-1, 1]$ in the uv-plane. In other words, $G(u, v)$ transforms D_0 into D.

We also have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$. Thus, $|\text{Jac}(G)| = \frac{1}{2}$. Therefore,

$$\begin{aligned}
\int\int_D (x+y)^2 e^{x^2-y^2} \, dA &= \int\int_{D_0} u^2 e^{uv} \, \frac{1}{2} du dv \\
&= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} u^2 e^{uv} \, dv du \\
&= \frac{1}{2} \int_{-1}^{1} u e^{uv} \Big|_{v=-1}^{v=1} \, du \\
&= \frac{1}{2} \int_{-1}^{1} u(e^u - e^{-u}) \, du \\
&= 2e^{-1},
\end{aligned}$$

the last step being a standard Calculus 2 problem that can be solved using integration by parts.

We then discussed how the previous problem is related to the notion of an inverse for $G(u, v)$ We knew how to express u, v in terms of x, y, but really wanted to express x, y in terms of u, v. What this means, is that we were given a function $F(x, y) = (u(x, y), v(x, y))$ that writes u, v in terms of x, y, and we want to "unravel" $F(x, y)$ so that we can express x, y in terms of u, v, via the function $G(u, v)$. This involves realizing $G(u, v)$ as the inverse of $F(x, y)$, or equivalently, regarding $F(x, y)$ as the inverse of $G(u, v)$. This lead to the:

Definition. The transformation $F(x, y) = (u(x, y), v(x, y))$ taking points in the xy-plane to points in the uv-plane is the *inverse of G(u,v)* if $F(G(u, v) = (u, v)$ for all (u, v) in the domain of G and $G(F(x, y)) = (x, y)$ for all (x, y) in the domain of F.

It is important that both these equations hold, otherwise F is not the inverse of G. By symmetry, it follows that G is the inverse of F. While it may not always be possible to find F given G, the idea is that if we express x and y in terms of u and v, we try to solve for u and v in terms of x and y to find F. Conversely, if we are given equations expressing u and v in terms of x and y, we regard this as F, and if we can solve for x and y in terms of u and v this gives G. We checked that in the example above $F(x,y) = (x+y, x-y)$ is the inverse of $G(u,v) = (\frac{u+v}{2}, \frac{u-v}{2})$. We also noted that if D is the original domain of integration in the xy-plane, and one knows the inverse $F(x,y)$, then $F(D) = D_0$ is the domain of integration in the uv-plane.

We followed this with a discussion explaining how the Jacobian comes into the change of variables formula for double integrals. In other words, why do we use $dA = |\frac{\partial(x,y)}{\partial(u,v)}|dudv$? To see this, we started with the transformation $G(u,v) = (x(u,v), y(u,v))$ and the following two facts:

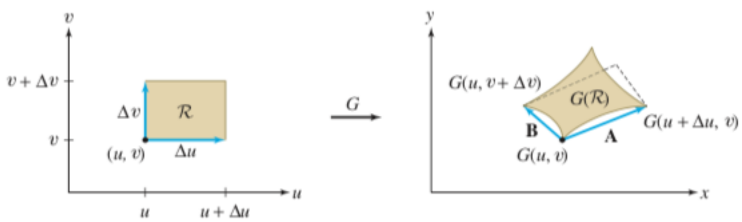
(i) If $A = a\vec{i} + b\vec{j}$ and $B = c\vec{i} + d\vec{j}$, then the area of the parallelogram spanned by A and B is the absolute value of $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = |ad - bc|$.

(ii) For a function $f(u,v)$ whose partial derivatives exist:

$$f(u+\Delta u, v) - f(u,v) \approx \Delta u \frac{\partial f}{\partial u} \text{ and } f(u, v+\Delta v) - f(u,v) \approx \Delta v \frac{\partial f}{\partial v},$$

when Δu and Δv are small. This follows, since for example, $\frac{\partial f}{\partial u} \approx \frac{f(u+\Delta u,v) - f(u,v)}{\Delta u}$.

Now, G transforms the rectangle with area $\Delta u \Delta v$ to the curvilinear rectangle shown below:



In the Riemann sums of the double integral in x and y over the region $G(\mathcal{R})$, we may use the parallelogram P spanned by spanned by the vectors $\mathbf{A}$ and $\mathbf{B}$ as small portions of area dA. Note that

$$\begin{aligned} \mathbf{A} &= (x(u+\Delta u, v) - x(u,v))\ \vec{i} + (y(u+\Delta u, v) - y(u,v))\ \vec{j} \\ &\approx \Delta u \frac{\partial x}{\partial u}\ \vec{i} + \Delta v \frac{\partial x}{\partial v}\ \vec{j}. \end{aligned}$$

Similarly: $\mathbf{B} \approx \Delta v \frac{\partial x}{\partial v}\ \vec{i} + \Delta v \frac{\partial y}{\partial v}\ \vec{j}$. Therefore:

$$\begin{aligned} dA &\approx \text{area}(\mathcal{R}) \\ &\approx \text{area}(P) \\ &\approx \left| \det \begin{pmatrix} \Delta u \frac{\partial x}{\partial u} & \Delta v \frac{\partial x}{\partial v} \\ \Delta u \frac{\partial y}{\partial u} & \Delta v \frac{\partial y}{\partial v} \end{pmatrix} \right| \\ &= \left| \Delta u \Delta v \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \Delta u \Delta v \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \\ &= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v. \end{aligned}$$

Thus, the Riemann sum: $\sum_i \sum_j f(x_i, y_j)\ dA$ in xy-coordinates is approximately the Riemann sum

$$\sum_i \sum_j f(x(u_j, v_j), y(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v,$$

in uv-coordinates. Passing to the limit as the units of area tend to zero gives the change of variables formula.

Wednesday, October 15. We began class by discussing improper double integrals, first recalling familiar cases for single improper integrals. As in the single variable case, we noted improper integrals occur either when the integrand is not defined on boundary points of the domain of integration, is unbounded on the domain of integration, or when the domain of integration is infinite. We illustrated these ideas with the following examples.

Example 1. $\int_0^1 \frac{1}{\sqrt{x}}\ dx$. Note that $f(x)$ is unbounded on $[0,1]$, but $\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\ dx = 2$ exists, so the original integral exists, or *converges* to 2.

Example 2. $\int_1^\infty e^{-2x}\ dx$. Note that the domain of integration is unbounded, but $\lim_{b\to\infty}\int_1^b e^{-2x}\ dx = \frac{1}{2e^2}$ exists, so the original integral converges.

Example 3. $\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA$, where $D: 0 \le x^2+y^2 \le 1$. Note that the integrand $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$ approaches infinity as points in the interior of the disk approach the circle $x^2+y^2=1$. Thus the integrand is unbounded on the domain of integration. Let D_a denote the disk $0 \le x^2+y^2 \le a^2$, with $0 < a < 1$. If the $\lim_{a\to 1}\int\int_{D_\epsilon} f(x,y)\ dA$ exists, then it will converge to the original integral, i.e.,

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = \lim_{a\to 1}\int\int_{D_a} \frac{1}{\sqrt{1-x^2-y^2}}\ dA.$$

We can use polar coordinates:

$$\begin{aligned}\int\int_{D_a} \frac{1}{\sqrt{-1-x^2-y^2}}\ dA &= \int_0^{2\pi}\int_0^a \frac{r}{\sqrt{1-r^2}}\ drd\theta \\ &= \int_0^{2\pi} -\sqrt{1-r^2}\Big|_{r=0}^{r=a}\ dA \\ &= \int_0^{2\pi} -\sqrt{1-a^2}+1\ dr \\ &= 2\pi\cdot(-\sqrt{1-a^2}+1).\end{aligned}$$

Taking the limit as $a\to 1$, we get 2π. Thus:

$$\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA = 2\pi.$$

In other words, $\int\int_D \frac{1}{\sqrt{1-x^2-y^2}}\ dA$ converges to 2π.

Example 4. $\int\int_D xye^{-x^2-y^2}\ dA$, where D is the first quadrant of $\mathbb{R}^2$.

Solution: In this case we can proceed as one might expect:

$$\int\int_D xye^{-x^2-y^2}\ dA = \lim_{a,b\to\infty}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy.$$

Here's one way to evaluate the iterated integral:

$$\begin{aligned}\int_0^b\int_0^a xye^{-x^2-y^2}\ dxdy &= \int_0^b\int_0^a (xe^{-x^2})(ye^{-y^2})\ dxdy \\ &= \int_0^b ye^{-y^2}(\int_0^a xe^{-x^2}\ dx)\ dy \\ &= (\int_0^b ye^{-y^2}\ dy)\cdot(\int_0^a xe^{-x^2}\ dx).\end{aligned}$$

Calculating these integrals separately:

$$\begin{aligned}\int_0^b ye^{-y^2}\ dy &= -\frac{1}{2}e^{-y^2}\Big|_{y=0}^{y=b} \\ &= \frac{1}{2}(-e^{-b^2}+1).\end{aligned}$$

Similarly: $\int_0^a xe^{-x^2}\,dx = \frac{1}{2}(-e^{-a^2}+1)$. Thus:

$$\int_0^b \int_0^a xye^{-x^2-y^2}\,dxdy = \frac{1}{2}(-e^{-b^2}+1)\cdot\frac{1}{2}(-e^{-a^2}+1).$$

Passing to the limit at $a \to \infty$ and $b \to \infty$, we get:

$$\int\int_D xye^{-x^2-y^2}\,dA = \frac{1}{4}.$$

We then calculated the improper integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ by calculating its square as a double integral over $\mathbb{R}^2$. We noted that the answer then yields $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}\,dx = 1$, a formula used when discussing normal distributions.

We then began a discussion of triple integrals.

A triple integral is an integral of the form $\int\int\int_B f(x,y,z)\,dV$, where B is a solid region contained in $\mathbb{R}^3$. The underlying idea for the definition of $\int\int\int_B f(x,y,z)\,dV$ is the same as we discussed for double integrals:

First, partition the domain of integration B into small subregions - in this case solids - of a similar type.

Second, select a point from each small subregion and evaluate the function $f(x,y,z)$ at that point.

Third, multiply the value obtained in the second step by the size of the subregion the point was chosen from. In this case we are multiplying the function value by a small unit of volume.

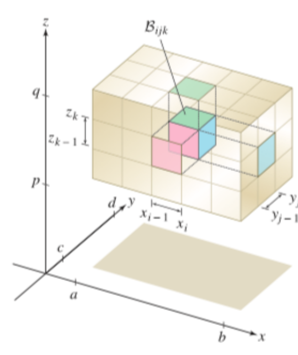
Fourth, add up the values from the previous step, thereby getting a Riemann sum.

Fifth, take a limit of the Riemann sums as the maximum volumes of the subregions in the partition go to zero.

Sixth, if the limit exists, we denote it $\int\int\int_B f(x,y,z)\,dV$.

As in previous discussions concerning double integrals, $\int\int\int_B f(x,y,z)\,dV$ is a quantity that depends upon $f(x,y,z)$ and the geometry of B and does not depend upon the coordinate system used to describe B or used to calculate $\int\int\int_B f(x,y,z)\,dV$.

Friday, October 17. we began class noting that we expect various versions Fubini's Theorem for triple integrals. If B is a rectangular box, and we use rectangular coordinates, our Riemann sums look something like this, which justifies Fubini's Theorem:



$$S_{N,M,L} = \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{L} f(P_{ijk})\,\Delta V_{ijk}$$

Fubini's Theorem for rectangular boxes. Suppose $B = [a,b] \times [c,d] \times [p,q]$ is a rectangular box in $\mathbb{R}^3$ and $f(x,y,z)$ is continuous on B. Then:

$$\int\int\int_B f(x,y,z)\ dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\ dzdydx.$$

Moreover, $\int\int\int_B f(x,y,z)\ dV$ can be calculated in any one of the five remaining ways to permute the order of integration. For example:

$$\begin{aligned}\int\int\int_B f(x,y,z)\ dV &= \int_c^d \int_p^q \int_a^b f(x,y,z)\ dxdzdy \\ &= \int_p^q \int_a^b \int_c^d f(x,y,z)\ dydxdz.\end{aligned}$$

Example 1. Calculate $\int\int\int_B x^2 + 2yz\ dV$, where $B = [0,1] \times [-1,0] \times [1,2]$.

Solution: Applying Fubini's Theorem,

$$\begin{aligned}\int\int\int_B x^2 + 2yz\ dV &= \int_0^1 \int_{-1}^0 \int_1^2 (x^2 + 2yz)\ dzdydx \\ &= \int_0^1 \int_{-1}^0 (x^2 z + yz^2)_{z=1}^{z=2}\ dydx \\ &= \int_0^1 \int_{-1}^0 (2x^2 + 4y) - (x^2 + y)\ dydz \\ &= \int_0^1 \int_{-1}^0 x^2 + 3y\ dydx \\ &= \int_0^1 (x^2 y + \frac{3}{2}y^2)_{y=-1}^{y=0}\ dx \\ &= \int_0^1 (x^2 - \frac{3}{2})\ dx \\ &= (\frac{x^3}{3} - \frac{3x}{2})_{x=0}^{x=1} \\ &= \frac{1}{3} - \frac{3}{2} = -\frac{7}{6}.\end{aligned}$$

Integrating in a different order we get:

$$\begin{aligned}\int\int\int_B x^2 + 2yz\ dV &= \int_{-1}^0 \int_1^2 \int_0^1 (x^2 + 2yz)\ dxdzdy \\ &= \int_{-1}^0 \int_1^2 (\frac{x^3}{3} + 2xyz)_{x=0}^{x=1}\ dzdy \\ &= \int_{-1}^0 \int_1^2 (\frac{1}{3} + 2yz)\ dzdy \\ &= \int_{-1}^0 (\frac{z}{3} + yz^2)_{z=1}^{z=2}\ dy \\ &= \int_{-1}^0 \frac{1}{3} + 3y\ dy \\ &= (\frac{y}{3} + \frac{3y^2}{2})_{y=-1}^{y=0} \\ &= 0 - (-\frac{1}{3} + \frac{3}{2}) = -\frac{7}{6}.\end{aligned}$$

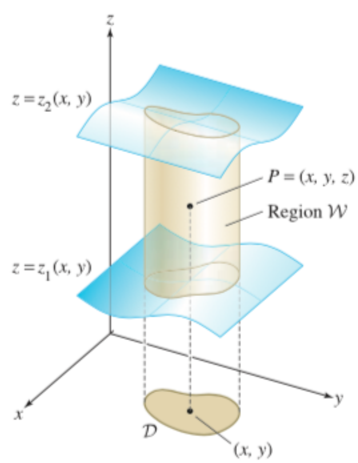
Fundamental Fact: $\int\int\int_B dV = \text{vol}(B)$. This follows, since in the Riemann sums approximating $\int\int\int_B dV$, we are just adding volumes of small solids covering B, so that each Reimann sum provides a better approximation to the volume of B and by passing to the limit, we obtain the volume of B.

Example 2. Let B denote the box in Example 1. Then:

$$\begin{aligned}\int\int\int_B dV &= \int_0^1 \int_{-1}^0 \int_1^2 dzdydx \\ &= \int_0^1 \int_{-1}^0 (2-1)\ dydx \\ &= \int_0^1 \int_{-1}^0 dydx \\ &= \int_0^1 (0-(-1))\ dx \\ &= \int_0^1 dx \\ &= 1. \\ &= \text{vol}(B)\end{aligned}$$

as expected.

What about more general domains of integration? Suppose we have a region $\mathcal{W} \subseteq \mathbb{R}^3$ defined as all points (x,y,z) in $\mathbb{R}^3$ such that $(x,y) \in D$ and $z_1(x,y) \leq z \leq z_2(x,y)$, as pictured below.

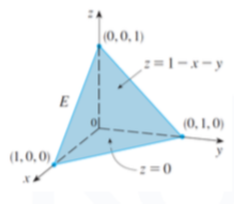


We have a corresponding version of Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f(x,y,z)\ dV = \int\int_D \{ \int_{z(_1x,y)}^{z_2(x,y)} f(x,y,z)\ dz \}\ dA.$$

Note that if we calculate the integral in the brackets by integrating with respect to z, and then evaluating z at $z_2(x,y)$ and $z_1(x,y)$ and subtracting, we then have a double integral of a function in x and y only over the domain $D \subseteq \mathbb{R}^2$. In this version of Fubini's Theorem, we can reduce a triple to a double integral. In the case above, we call W a z-simple region.

Example 3. Calculate $\int\int\int_B e^{x+y+z}\ dV$ for B the solid tetrahedron below:



Solution. Regarding B as a z-simple region, we have that $0 \leq z \leq 1-x-y$ and D is the triangle $0 \leq x \leq 1$, $0 \leq y \leq 1-x$. Thus:

$$\begin{aligned}
\int\int\int_B e^{x+y+z}\ dV &= \int\int_D \left\{ \int_0^{1-x-y} e^{x+y+z}\ dz \right\} dA. \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{x+y+z}\ dzdydx \\
&= \int_0^1 \int_0^{1-x} e^{x+y+z}\Big|_{z=0}^{z=1-x-y} dydx \\
&= \int_0^1 \int_0^{1-x} (e^1 - e^{x+y})\ dydx \\
&= \int_0^1 (ey - e^{x+y})_{y=0}^{y=1-x}\ dx \\
&= \int_0^1 (e(1-x) - e) - (0 - e^x)\ dx \\
&= \int_0^1 (-ex + e^x)\ dx \\
&= (-\frac{e}{2}x^2 + e^x)_0^1 \\
&= (-\frac{e}{2} + e) - (0+1) \\
&= \frac{e}{2} - 1.
\end{aligned}$$

We then noted the following. Suppose B is a z-simple region defined over the domain $D \subseteq \mathbb{R}^2$. If D is a region of Type 1, say $c(x) \leq y \leq d(x)$, $a \leq x \leq b$, then

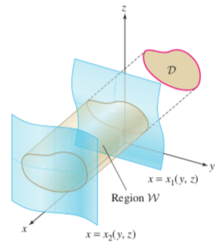
$$\int\int\int_{\mathcal{B}} f(x,y,z)\ dV = \int_a^b \int_{c(x)}^{d(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz\ dy\ dx,$$

while if D is a region of Type 3 described as: $a(y) \leq x \leq b(y)$ and $c \leq y \leq d$, then

$$\int\int\int_{\mathcal{B}} f(x,y,z)\ dV = \int_c^d \int_{a(y)}^{b(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\ dz\ dx\ dy.$$

We also noted that the one may have x-simple and y-simple regions, where in the first case, B is bounded in the front and back by functions $x = a(y,z)$ and $x = b(y,z)$, while in the second case, B is bounded to the left and right by functions $y = c(x,z)$ and $y = d(x,z)$, with the usual orientation of the x, y, z-axes. In our MW text, the z-simple, x-simple, and y-simple regions are called regions of Type I, II, or III, respectively.

To elaborate, nn x-simple solid has the form $x_1(y,z) \leq x \leq x_2(y,z)$, $(y,z) \in D$, with D in the yz-plane.



By Fubini's Theorem:

$$\int\int\int_{\mathcal{W}} f\ dV = \int\int_{D} \{ \int_{x_1(y,z)}^{x_2(y,z)} f\ dx \}\ dA.$$

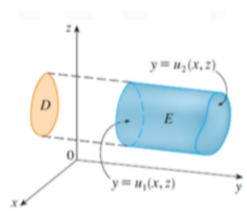
If D is described as: $a(z) \leq y \leq b(z)$ and $c \leq z \leq d$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_c^d \int_{a(z)}^{b(z)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxdydz.$$

While if D is described as: $c(y) \leq z \leq d(y)$ and $a \leq y \leq b$, then

$$\int\int\int_{\mathcal{W}} f\ dV = \int_a^b \int_{c(y)}^{d(y)} \int_{x_1(y,z)}^{x_2(y,z)} f\ dxddzdy.$$

A y-simple solid E has the form $u_1(x,z) \leq y \leq u_2(x,z)$, $(y,z) \in D$, with D in the xz-plane.



By Fubini's Theorem:

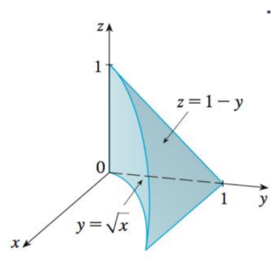
$$\int\int\int_{E} f\ dV = \int\int_{D} \{ \int_{u_1(x,z)}^{u_2(x,z)} f\ dx \}\ dA.$$

which can take the form

$$\int_c^d \int_{a(x)}^{b(x)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydzdx \quad \text{or} \quad \int_a^b \int_{c(z)}^{d(z)} \int_{u_1(x,z)}^{u_2(x,z)} f\ dydxdz$$

We then calculated the following example in two ways, first as a z-simple region, then as an x-simple region.

Example 1. Calculate $\int\int\int_B x\ dV$ for B



Solution

$$\begin{aligned}
\int\int\int_B x\ dV &= \int\int_D \{\int_0^{1-y} x\ dz\}\ dA \\
&= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} x\ dzdydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 xz\Big|_{z=0}^{z=1-y}\ dydx \\
&= \int_0^1 \int_{\sqrt{x}}^1 x(1-y)\ dydx \\
&= \int_0^1 x(y-\frac{y^2}{2})_{y=\sqrt{x}}^{y=1}\ dx \\
&= \int_0^1 x(\frac{1}{2}-\sqrt{x}+\frac{x}{2})\ dx \\
&= (\frac{x^2}{4}-\frac{2}{5}x^{\frac{5}{2}}+\frac{x^3}{6})_0^1 \\
&= \frac{1}{4}-\frac{2}{5}+\frac{1}{6} \\
&= \frac{1}{60}
\end{aligned}$$

To calculate Calculate $\int\int\int_B x\ dV$ for B above, viewed as a x-simple surface, we first notice that if we move inside B in the direction of the x-axis, we get $0 \le x \le y^2$. The projection of B onto the yz-plane is the area in the first quadrant below the line $z = 1-y$. Thus, we have

$$\begin{aligned}
\int\int\int_B x\ dV &= \int_0^1 \int_0^{1-y} \int_0^{y^2} x\ dx\ dz\ dy \\
&= \int_0^1 \int_0^{1-y} \{\frac{x^2}{2}\}_{x=0}^{x=y^2}\ dz\ dy \\
&= \int_0^1 \int_0^{1-y} \frac{y^4}{2}\ dz\ dy \\
&= \int_0^2 \{\frac{y^4}{2}z\}_{z=0}^{z=1-y}\ dy \\
&= \int_0^1 \frac{y^4}{2}-\frac{y^5}{2}\ dy \\
&= \frac{1}{10}-\frac{1}{12} \\
&= \frac{1}{60}.
\end{aligned}$$

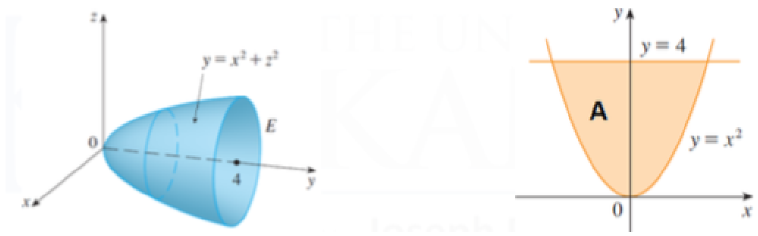
Monday, October 20. We began class by stating the two facts:

(i) $\int\int\int_B\ dV = \text{volume}(B)$.
(ii) The average value of $f(x,y,z)$ over B is: $\frac{1}{\text{vol}(B)}\int\int\int_B f(x,y,z)\ dV$.

We verified the volme formula by using a triple integral to show that the volme of a cylinder with radis r and height h is $\pi r^2 h$.

We then worked the following example:

Example. This example shows how changing the order of integration can simplify the integration. Consider $\int\int\int_E \sqrt{x^2+z^2}\ dV$ for the solid E pictured on the left

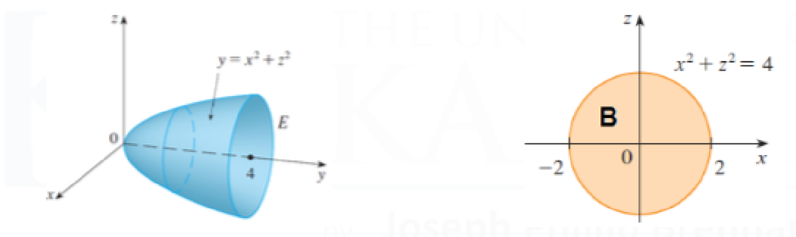


Regarding E as a z-simple region, with domain of integration A, we have

$$\begin{aligned}\int\int\int_E \sqrt{x^2+z^2}\ dV &= \int\int_A \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dz\ dA\\ &= \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdyx\\ &= \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\ dzdxdy,\end{aligned}$$

which is difficult to integrate

On the other hand, regarding E as a y-simple region with domain of integration , we tegrate with respect to y first, to obtain



$$\begin{aligned}\int\int\int_E \sqrt{x^2+z^2}\ dV &= \int\int_B \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\ dy\ dB\\ &= \int\int_B 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\ dB\\ &= \int_0^{2\pi}\int_0^2 (4r-r^3)r\ drd\theta\\ &= 2\pi\int_0^2 4r^2 - r^4\ dr\\ &= 2\pi(\frac{32}{3}-\frac{32}{5})\\ &= \frac{128\pi}{15}.\end{aligned}$$

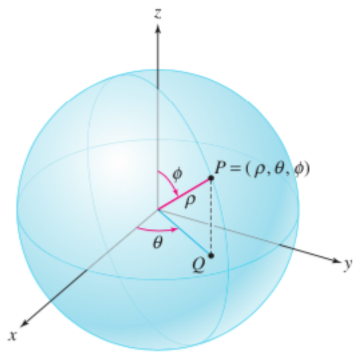
We ended class by considering the integral $\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV$, where B is the solid sphere of radius one centered at the origin. We noted that this requires a three dimensional version of polar coordinates, and suggested that a substitution using spherical coordinates should be the analogous procedure.Using the substitutions $x = \rho\cos(\theta)\sin(\phi), y = \rho\sin(\theta)\sin(\phi), z = \rho\cos(\phi), dV = \rho^2\sin(\phi)$, with limits of integration

$0 \le \rho \le 1, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$, we saw that

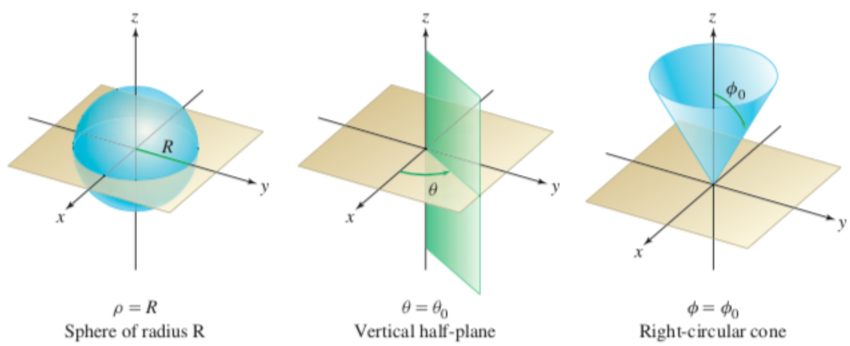
$$\int\int\int_B \sqrt{x^2+y^2+z^2}\ dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho \cdot \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta = \pi.$$

This led to the interesting conclusion that the average distance from the origin for points in B is $\frac{3}{4}$.

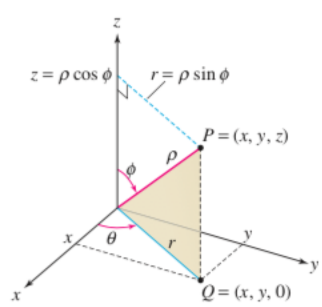
Wednesday, October 22. We began class by reviewing the definition of spherical coordinates. One begins by observing that every point in $\mathbb{R}^3$ lies on a sphere of radius ρ centered at (0,0,0) and thus can be expressed in terms of spherical coordinates, ρ, ϕ, θ. Here is typical point P using spherical coordinates



Note: $P = (\rho, \phi, \theta)$, with $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. If we set each of the spherical coordinate equal to a constant, we get:



We then showed the relation between the spherical coordinates and the rectangular coordinates of P.



Note:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta), y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi).$$

It follows from these equations that the expression $x^2+y^2+z^2$, in spherical coordinates, becomes ρ^2. We will use this fact often. Writing spherical coordinates in terms of rectangular coordinates.

$\rho = \sqrt{x^2+y^2+z^2}$

$\tan(\theta) = \frac{y}{x}$, so $\theta = \tan^{-1}(\frac{y}{x})$.
$\cos(\phi) = \frac{z}{\rho}$, so $\phi = \cos^{-1}(\frac{z}{\rho})$.

We then stated

Fubini's Theorem for spherical coordinates. Suppose B in spherical coordinates B is the set of points satisfying satisfies $\rho_1(\phi, \theta) \leq \rho \leq \rho_2(\phi, \theta)$, $\phi_1 \leq \phi \leq \phi_2$, and $\theta_1 \leq \theta \leq theta_2$. Then

$$\int\int\int_B f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)) \cdot \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta.$$

We then considered the following

Example. Find the volume of the region B bounded below by the cone $z = \sqrt{3(x^2+y^2)}$ and above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$. The key point in setting up this integral in spherical coordinates is to notice that the equation of a cone in spherical coordinates is $z = \phi_0$ for some angle $\phi = \phi_0$. In this case, $z = \sqrt{3(x^2+y^2)}$ in spherical coordinates becomes

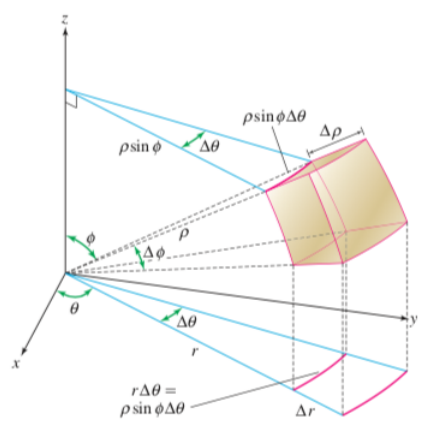
$$\rho\cos(\phi) = \sqrt{3(\rho^2 cos^2(\theta)\sin^2(\phi) + \rho^2\sin^2(\theta)\sin^2(\theta))} = \rho\sqrt{3}\sin(\phi).$$

Thus, $\tan(\phi) = \frac{1}{\sqrt{3}}$, and thus, $\phi = \frac{\pi}{6}$. Therefore,

$$\text{vol}(B) = \int\int\int_B\ dV = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2\sin(\phi)\ d\rho\ d\phi\ d\theta,$$

which can easily be worked out.

We then briefly discussed why we use $dV = \rho^2\sin(\phi)\ d\rho d\phi d\theta$ in the conversion to spherical coordinates by looking at the diagram below.



Recall that $r = \rho\sin(\phi)$. The top face of the spherical cube above has dimensions $(\rho\sin(\phi)\Delta\theta) \times \Delta\rho$. The remaining edge (subtending an angle $\Delta\phi$) of the spherical cube is approximately $\rho\Delta\phi$. The volume of the spherical cube is approximately:

$$(\rho\sin(\phi)\Delta\theta) \cdot (\Delta\rho) \cdot (\rho\Delta\phi) = \rho^2\sin(\phi)\Delta\rho\Delta\phi\Delta\theta.$$

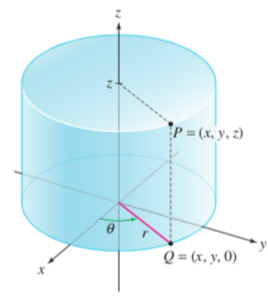
Since once can use small spherical wedges like above as a partition of B in the Riemann sum for $\int\int\int_B f(x,y,z)\ dV$, it follows that the Riemann sum in spherical coordinates takes the form

$$\sum_i\sum_j\sum_k f(\rho_i\sin(\phi_j)\cos(\theta_k), \rho_i\sin(\phi_j)\sin(\theta_k), \rho_i\cos(\phi_j)) \cdot \rho^2\sin(\phi)\Delta\rho\Delta\phi\Delta\theta,$$

so that taking the limit as the volumes tend to zero gives the formula for $\int\int\int_B f(x,y,z)\ dV$ in spherical coordinates.

We then noted that every point in $\mathbb{R}^3$ lies on the top edge of a cylinder, which enables us to describe points in $\mathbb{R}^3$ in terms of cylindrical coordinates, noting that cylindrical coordinates are essentially like polar coordinates, though with the extra variable z.

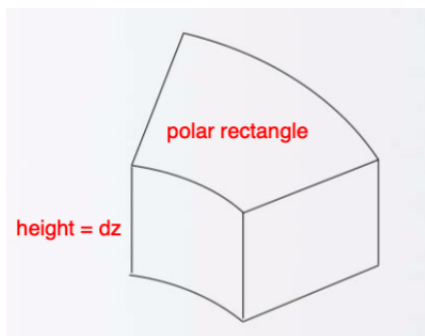
In cylindrical coordinates, $P = (r, \theta, z)$.



To transform a triple integral into cylindrical coordinates, we set:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z, \quad dV = r\ dz\ dr\ d\theta.$$

We can easily guess what the volume of a cylindrical wedge should be.



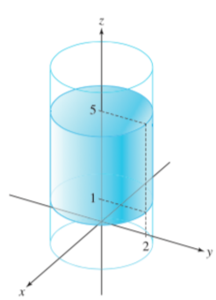
It should be approximately the area of the corresponding polar rectangle times Δz, the change in the z direction, i.e., the height of the cylindrical wedge. Thus, in cylindrical coordinates, small units of volume dV are approximately $(r\Delta r\Delta\theta)\cdot\Delta z = r\ \Delta z\ \Delta r\ \Delta\theta$. Using expressions of this type in the Riemann sum for $\int\int\int_B f(x,y,z)\ dV$ in cylindrical coordinates, we get the following version of Fubini's theorem:

Fubini's Theorem in Cylindrical Coordinates. Given a bounded region $B \subseteq \mathbb{R}^3$ described in cylindrical coordinates as: $z_1(r,\theta) \le z_2(r,\theta); r_1(\theta) \le r \le r_r(\theta); \theta_1 \le \theta \le \theta_2$, and $f(x,y,z)$ continuous on B, we have:

$$\int\int\int_B f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z)\ r\ dz\ dr\ d\theta.$$

We then worked some examples using cylindrical coordinates.

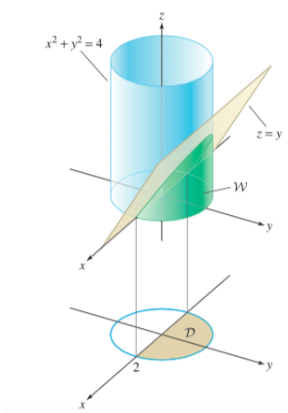
Example 2. Integrate $z\sqrt{x^2+y^2}$ over the cylinder B given below.



Solution: In cylindrical coordinates $B : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 1 \leq z \leq 5$. Thus,

$$\begin{aligned}
\int\int\int_B z\sqrt{x^2+y^2} &= \int_0^{2\pi}\int_0^2\int_1^5 z\sqrt{(r\cos(\theta))^2+(r\sin(\theta))^2}\ r\ dzdrd\theta \\
&= \int_1^5\int_0^{2\pi}\int_0^2 zr^2\ drd\theta dz \\
&= 2\pi\int_1^5\int_0^2 zr^2\ drdz \\
&= 2\pi\int_1^5 z(\frac{2^3}{3}-0)\ dz \\
&= \frac{16\pi}{3}\int_1^5 z\ dz \\
&= \frac{16\pi}{3}\{\frac{5^2}{2}-\frac{1}{2}\} \\
&= 64\pi.
\end{aligned}$$

Example 3. Calculate $\int\int\int_W z\ dV$ for W bounded by the cylinder $0 \leq x^2+y^2 \leq 4$ and the planes $z = y$, with $y \geq 0$.



To describe W is cylindrical coordinates: $0 \leq z \leq y$, so $0 \leq z \leq r\sin(\theta)$. Since $y \geq 0$, the projection of W onto the xy-plane is the semi-circle D. Thus, $0 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$\begin{aligned}
\int\int\int_W z\ dV &= \int_0^{\pi}\int_0^2\int_0^{r\sin(\theta)} z\ rdzdrd\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^2 r^3\sin^2(\theta)\ drd\theta \\
&= \frac{1}{2}\int_0^{\pi}\frac{16}{4}\sin^2(\theta)\ d\theta \\
&= 2\int_0^{\pi}\frac{1}{2}-\frac{1}{2}\cos(2\theta)\ d\theta \\
&= 2\{\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\}_0^{\pi} \\
&= \pi.
\end{aligned}$$

Friday, October 14. The class worked on practice problems for Exam 2.

Monday, October 27. The class turned in their Math Diaries and worked on practice problems for Exam 2.

Wednesday, October 29. We briefly discussed the change of variables theorem for triple integrals. For this we needed to define the determinant of a 3×3 matrix. We noted that there are several ways to do this. With this in hand, we stated

Change of variables theorem for triple integrals. Suppose $G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$ is a transformation from the uvw-coordinate system to the xyz-coordinate system such that $G(u, v, w)$ is one-to-one and its coordinate functions have continuous first order partial derivatives. Suppose $G(B_0) = B$, where B_0 is a solid in the uvw-coordinate system and B is a solid in the xyz-coordinate system. If $f(x, y, z)$ is continuous on B, then

$$\int\int\int_B f(x,y,z)\ dV = \int\int\int_{B_0} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ dV,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_y & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

is the Jacobian of $G(u, v, w)$. Just as in the two variable vase, small units of volume in the xyz-coordinate system correspond to $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ times small units of volume in the uvw-coordinate system.

We gave the following examples:

(i) $T(u, v, w) = (au + dv + gw, bu + ev + hw, cu + fv + iw)$ takes the unit cube in the uvw-coordinate system spanned by the unit vectors e_1, e_2, e_3 to the parallelopiped in the xyz-coordinate system spanned by the vectors $v_1 = (a, b, c), v_2 = (d, e, f), v_3 = (g, h, i)$.

(ii) $H(u, v, w) = (u + \alpha, v + \beta, c + \gamma)$ translates vectors with endpoints at (0,0,0) in the uvw-coordinate system to vectors with endpoints (α, β, γ) in the xyz-coordinate system.

(iii) Spherical coordinates transform boxes in the (ρ, θ, ϕ)-coordinate system to spheres in the xyz-coordinate system.

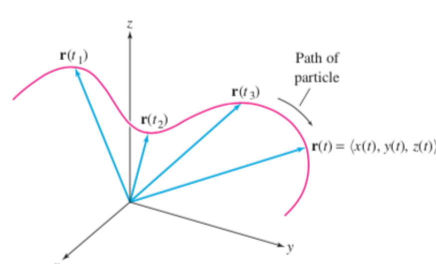
(iv) Cylindrical coordinates transform boxed in the (r, θ, z)-coordinate system to cylinders in the xyz-coordinate system.

Friday, October 31. Our next goal is to integrate along a curve in $\mathbb{R}^3$. For this, we began a discussion of vector valued functions.

Definition. A vector valued function is a function

$$\mathbf{r}(t) = x(t)i + y(t)j + z(t)k = (x(t), y(t)z(t)),$$

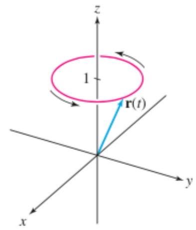
with t belonging to a subset of $\mathbb{R}$.



If the values of $\mathbf{r}(t)$ lie in the xy-plane, we write

$$\mathbf{r}(t) = (x(t), y(t)) = x(t)i + y(t)j.$$

The variable t is called the *parameter*. It is often convenient to think of t as time. The set of points traced out by $\mathbf{r}(t)$ is a *curve*. How the curve is traced out is called a *path*. For example, for the curve C, the circle of radius one, with center $(0, 0, 1)$



one can have several different paths:

$\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 2\pi$ traces the curve once.

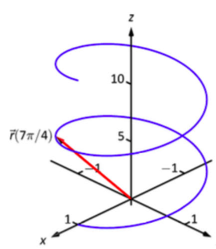
$\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \le t \le \pi$ traces once, but twice as fast.

$\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.

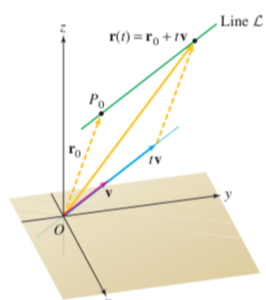
$\mathbf{r}_4(t) = (\cos(2\pi - t), \sin(2\pi - t), 1)$, traces the curve once, in the opposite direction of $\mathbf{r}_1(t)$.

The different paths $\mathbf{r}_i(t)$ above are referred to as different *parametrizations* of C. We then gave two more examples of curves in $\mathbb{R}^3$.

Example. The helix $\mathbf{r}(t) = (cos(t), \sin(t), t)$, $t \ge 0$.



Example. The line through a point $P_0 = (a, b, c)$ parallel to a given vector $\vec{v} = v_1 i + v_2 j + v_3 k$.



(B) Line through P_0 in the direction of $\mathbf{v}$

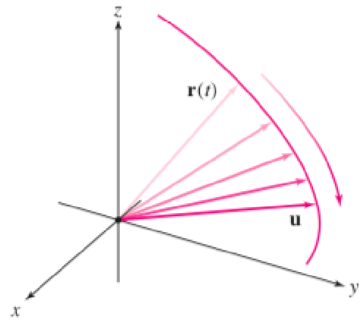
$$\mathbf{r}(t) = P_0 + t\vec{v} = (a + tv_1, b + tv_2, c + tv_3).$$

We then noted that limits and continuity for vector valued functions are defined in a similar way as for functions we have previously encountered.

Limits. For a vector valued function $\mathbf{r}(t)$ and a fixed vector $\vec{u} = u_1 i + u_2 j + u_3 k$, we write:

$$\lim_{t \to t_0} \mathbf{r}(t) = \vec{u} \quad \text{if} \quad \lim_{t \to t_0} ||\mathbf{r}(t) - \vec{u}|| = 0.$$

Note that this means the vectors $\mathbf{r}(t)$ get closer to the vector $\vec{u}$ as t approaches t_0.



DF **FIGURE 1** The vector-valued function $\mathbf{r}(t)$ approaches the vector $\mathbf{u}$ as $t \to t_0$.

This is equivalent to

$$\begin{aligned}\lim_{t\to t_0} x(t) &= u_1\\ \lim_{t\to t_0} y(t) &= u_2\\ \lim_{t\to t_0} z(t) &= u_3.\end{aligned}$$

To explain why this holds, we noted

$$\begin{aligned}\lim_{t\to t_0} ||\mathbf{r}(t)-\vec{u}|| &= \lim_{t\to t_0}\sqrt{(x(t)-u_1)^2+(y(t)-u_2)^2+(z(t)-u_3)^2}\\ &= \sqrt{(\lim_{t\to t_0} x(t)-u_1)^2+(\lim_{t\to t_0} y(t)-u_2)^2+(\lim_{t\to t_0} z(t)-u_3)^2}\end{aligned}$$

by continuity of the square root and square functions. If

$$\begin{aligned}\lim_{t\to t_0} x(t) &= u_1\\ \lim_{t\to t_0} y(t) &= u_2\\ \lim_{t\to t_0} z(t) &= u_3,\end{aligned}$$

then the limits under the radical are zero. Thus, $\lim_{t\to t_0} ||\mathbf{r}(t)-\vec{u}|| = 0$. i.e., $\lim_{t\to t_0}\mathbf{r}(t)=\vec{u}$.

Example. If $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, then

$$\begin{aligned}\lim_{t\to\pi}\mathbf{r}(t) &= (\lim_{t\to\pi} 5\cos(t), \lim_{t\to\pi} -3\sin(\frac{t}{2}), \lim_{t\to\pi} e^{3t+4})\\ &= (5\cos(\pi), -3\sin(\frac{\pi}{2}), e^{3\pi+4})\\ &= (-5, -3, e^{3\pi+4}).\end{aligned}$$

Continuity. $\mathbf{r}(t)$ is *continuous at* t_0 if $\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{r}(t_0)$. From the discussion of limits, we observed that this is equivalent to:

$$x(t), y(t), z(t) \text{ are all continuous at } t_0.$$

We then noted that differentiabiility is defined as expected.

Differentiability. $\mathbf{r}(t)$ is *differentiable* at t_0 if

$$\lim_{t\to h}\frac{1}{h}\cdot\{\mathbf{r}(t+h)-\mathbf{r}(t_0)\},$$

exists. Note this limit involves a scalar times a vector, and is a vector if the limit exists. If the limit exists, we write it as $\mathbf{r}'(t_0)$ or $\frac{d}{dt}\mathbf{r}(t)|_{t_0}$.

Fact: $\mathbf{r}'(t)$ is differentiable at t_0 exactly when $x(t), y(t), z(t)$ are all differentiable at t_0, in which case

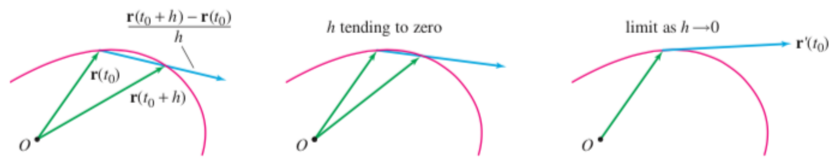
$$\mathbf{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$

This follows, since we may take limits coordinate-wise (and because the scalar $\frac{1}{h}$ can be moved inside of an ordered triple):

$$\lim_{t\to h} \frac{1}{h} \cdot \{\mathbf{r}(t+h) - \mathbf{r}(t_0)\} = (\lim_{t\to h} \frac{x(t_0+h)-x(t_0)}{h}, \lim_{t\to h} \frac{y(t_0+h)-y(t_0)}{h}, \lim_{t\to h} \frac{z(t_0+h)-z(t_0)}{h})$$
$$= (x'(t_0), y'(t_0), z'(t_0)).$$

Example. Given $\mathbf{r}(t) = (5\cos(t), -3\sin(\frac{t}{2}), e^{3t+4})$, $\mathbf{r}'(t) = (-5\sin(t), -\frac{3}{2}\cos(\frac{t}{2}), 3e^{3t+4})$. Therefore: $\mathbf{r}\,'(\pi) = (-5\sin(\pi), \frac{3}{2}\cos(\frac{\pi}{2}), e^{3\pi+4}) = (0, 0, e^{3\pi+4})$.

We finished class by noting that the vector $\mathbf{r}'(t_0)$ is tangent to the curve $\mathbf{r}(t)$ at the point $P_0 = \mathbf{r}(t_0)$, as illustrated in the diagram below.



We then noted that we have versions of the familiar rules of differentiation for vector valued functions.

Properties of the derivative. Assuming differentiablity:

(i) $(\mathbf{r}(t) + \mathbf{r}s(t))' = \mathbf{r}'(t) + \mathbf{r}s'(t)$.
(ii) $(\lambda\mathbf{r}(t))' = \lambda\mathbf{r}'(t)$, for $\lambda \in \mathbb{R}$.
(iii) $(f(t)\cdot\mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$, for $f(t)$ a scalar valued function.
(iv) $\mathbf{r}(g(t))' = g'(t)\mathbf{r}'(g(t))$, for $g(t)$, a scalar function.
(v) $(\mathbf{r}(t)\cdot\vec{s}(t))' = \mathbf{r}'(t)\cdot\vec{s}(t) + \mathbf{r}(t)\cdot\vec{s}'(t)$.
(vi) $(\mathbf{r}(t)\times\vec{s}(t))' = \mathbf{r}\,'(t)\times\vec{s}(t) + \mathbf{r}(t)\times\vec{s}\,'(t)$.

Monday, November 3. We continued our discussion of curves in $\mathbb{R}^3$ or $\mathbb{R}^2$ by considering the length of a path or curve.

Definition. Suppose $\mathbf{r}(t)$ is differentiable and $\mathbf{r}'(t)$ is continuous on $[a, b]$. Then the length of the path from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ is given by

$$s = \int_a^b ||\mathbf{r}'(t)||\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

The parameter s is called *arc length*. We can keep track of the arc length as we move along the path by considering the function:

$$s(t) = \int_a^t ||\mathbf{r}'(u)||\, du,$$

for $a \le t \le b$. We noted that s gives the length of the path $\mathbf{r}(t)$ for the range $a \le t \le b$, and that this is also the length of the corresponding curve, under an additional hypothesis.

IMPORTANT POINTS. (a) If $\mathbf{r}(t)$ is 1-1, then the arc length of the path equals the length of the curve traced out by $\mathbf{r}(t)$.

(b) The length of the curve traced out by $\mathbf{r}(t)$ is *independent of the parametrization.*

Example. We then revisited the examples from the previous lecture of the circle of radius one centered at (0,0,1) and its four parametrizations.

(i) $\mathbf{r}_1(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 2\pi$ traces the curve once.
(ii) $\mathbf{r}_2(t) = (\cos(2t), \sin(2t), 1)$, with $0 \le t \le \pi$ traces once, but twice as fast.
(ii) $\mathbf{r}_3(t) = (\cos(t), \sin(t), 1)$, with $0 \le t \le 4\pi$, traces the curve twice, but at the same speed at $\mathbf{r}_1(t)$.

(iv) $\mathbf{r}_4(t) = (\cos(2\pi - t), \sin(2\pi - t), 1)$, traces the curve, once in reverse order.

We expect the lengths of (i), (i), (iv) to be 2π and the length of the path (iii) to be 4π.

For $\mathbf{r}_1(t) : ||\mathbf{r}_1'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus:

$$s = \int_0^{2\pi} ||\mathbf{r}_1'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

Note that $\mathbf{r}_1(t)$ is 1-1 on $[0, 2\pi]$, so this gives the expected length of the curve.

For $\mathbf{r}_2(t) : ||\mathbf{r}_2'(t)|| = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2 + 0} = 2$. Thus:

$$s = \int_0^{\pi} ||\mathbf{r}_2'(t)||\ dt = \int_0^{\pi} 2\ dt = 2\pi.$$

Note that $\mathbf{r}_2(t)$ is also 1-1, so we get that the length of the path equals the length of the curve. Note also that the previous two parametrizations are the different, but yield the same arc length.

For $\mathbf{r}_3(t) : ||\mathbf{r}_3'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 0} = 1$. Thus

$$s = \int_0^{4\pi} ||\mathbf{r}_3'(t)||\ dt = \int_0^{4\pi} 1\ dt = 4\pi.$$

Note here that the path traces the curve twice, so the length of the path is 4π, while the length of the curve is 2π. In this case, $\mathbf{r}_3(t)$ is NOT 1-1 on the interval $[0, 4\pi]$, which explains why the length of the path differs from the length of the curve.

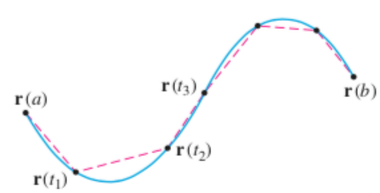
For $\mathbf{r}_4(t) : ||\mathbf{r}_4'(t)|| = \sqrt{(\cos(2\pi - t))^2 + (-\sin(2\pi - t))^2 + 0^2} = 1$. Thus

$$s = \int_0^{\pi} ||\mathbf{r}_4'(t)||\ dt = \int_0^{2\pi} 1\ dt = 2\pi,$$

as expected.

We then noted that if we think of t as time, then $s(t) = \int_a^t ||\mathbf{r}'(t)||\ dt$ gives the distance traveled in time t along the path described by $\mathbf{r}(t)$, so that $s'(t) = ||\mathbf{r}'(t)||$ denotes the speed. Thus, the *velocity vector* $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve (since it is tangent to the curve) and the length of the velocity vector gives the speed at time t.

We then gave a heuristic description accounting for the formula for the arc length along a segment of a curve, say C is given by $\mathbf{r}(t)$, with $a \leq t \leq b$. Partition the interval $[a, b] : a = t_1 < t_1 \cdots < t_n = b$ so that $t_{i+1} - t_i = \Delta t$ is small.



Create a polygonal path whose endpoints are the $\mathbf{r}(t_i)$. The length of each line segment in the path is $||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)||$. We have:

$$||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| = \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2}$$

For small values of Δt:

$$x(t_{i+1}) - x(t_i) \approx x'(t_i)\Delta t$$
$$y(t_{i+1}) - y(t_i) \approx y'(t_i)\Delta t$$
$$z(t_{i+1}) - z(t_i) \approx z'(t_i)\Delta t$$

Thus:

$$\begin{aligned}
||\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)|| &= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2} \\
&\approx \sqrt{(x'(t_i)\Delta t)^2 + (y'(t_i)\Delta t)^2 + (z'(t_i)\Delta t)^2} \\
&= \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t \\
&= ||\mathbf{r}'(t_i)||\ \Delta t
\end{aligned}$$

Summing these expressions we get an approximation of the arc length on the one hand, and a Riemann sum for $\int_a^b ||\mathbf{r}'(t)||\ dt$ on the other hand. Passing to the limit as $\Delta t \to 0$ gives the arc length formula.

We then turned to the definition of $\int_C f(x, y, z)\ ds$, the line integral of the scalar function $f(x, y, z)$ along the curve C. We proceed as before, as with all previous types of integration.

Step 1. Subdivide C into finitely many smaller curves C_i of the same length Δs.

Step 2. Choose a point (x_i, y_i, z_i) from the component C_i.

Step 3. Multiply $f(x_i, y_, z_i)$ by the size of each C_i to get $f(x_i, y_i, z_i)\ \Delta s$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta s$.

Step 5. Take the limit of the Riemann sums as $\Delta s \to 0$, to get:

$$\int_C f(x, y, z)\ ds,$$

the *line (or path) integral over* $f(x, y, z)$ *over* C. This integral is sometimes called the *the line integral of* $f(x, y, z)$ *with respect to arc length*.

We must use the parametrization $\mathbf{r}(t)$ of C to calculate $\int_C f(x, y, z)\ ds$. From our discussion of arc length, we have that small portions of length Δs along the curve are approximated by $||\mathbf{r}'(t_i)||\Delta t$, where $(x_i, y_i, z_i) = \mathbf{r}(t_i)$. Now use

$$\Sigma_i f(x_i, y_i, z_i)||\mathbf{r}'(t_i)||\ \Delta t$$

in the Riemann sum (Step 4) above. Taking the limit as $t \to 0$ we get:

$$\begin{aligned}
\int_C f(x, y, z)\ ds &= \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||\ dt \\
&= \int_a^b f(x(t), y(t), z(t))||\mathbf{r}'(t)||\ dt.
\end{aligned}$$

A word of caution: Depending upon the context, one will either calculate a *path integral*, if the parametrization is given or one may have to choose a parametrization to calculate a line integral. Moreover, if the path $\mathbf{r}(t)$ doubles back on itself, or repeats portions of the curve with non-zero length, this will be reflected in $\int_a^b f(\mathbf{r}(t))\ ||\mathbf{r}'(t)||\ dt$. Note that:

$$\int_C\ ds = \int_a^b ||\mathbf{r}'(t)||\ dt,$$

is the arc length of C, which is what we expect.

We then worked the following example
Example. Calculate $\int_C (x^2 + y^2)z^3\ ds$, for C that part of the helix $\mathbf{r}(t) = (\cos(t), \sin(t), t)$, for $0 \le t \le \frac{\pi}{4}$. .

Solution. $\mathbf{r}'(t) = (-s \in)t_, \cos(t), 1)$, so $||\mathbf{r}'(t)|| = \sqrt{2}$.

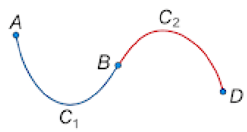
Note that $\mathbf{r}'(t) = (\sin(t), \cos(t), 1)$, so that $||\mathbf{r}'(t)|| = \sqrt{2}$

$$\begin{aligned}\int_C (x^2+y^2)z^3 \; ds &= \int_0^{\frac{\pi}{4}} (\cos^2 t + \sin^2 t)t^3\sqrt{2} \; dt \\ &= \frac{\sqrt{2}}{4} t^4 \Big|_0^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}\pi}{1024}.\end{aligned}$$

Wednesday, November 5. We began class by discussing the following properties of line integrals.

Properties of line integrals. Assuming the line integrals below exist, we have

(i) $\int_C f(x,y,z) + g(x,y,z) \; ds = \int_C f(x,y,z) \; ds + \int_C g(x,y,z) \; ds$.
(ii) $\int_C \lambda f(x,y,z) \; ds = \lambda \int_C f(x,y,z) \; ds$, $\lambda \in \mathbb{R}$.
(iii) $\int_C f(x,y,z) \; ds = \int_{C_1} f(x,y,z) \; ds + \int_{C_2} f(x,y,z) \; ds$.



(iv) $\int_C f(x,y,z) \; ds$ is independent of the parametrization, if the parametrization is 1-1 with continuous derivative.

Example. This example illustrates property (iii). Let C be the triangle in $\mathbb{R}^3$ with vertices (2,0,0), (0,2,0), (0,0,2). The C has three components, C_1, the line segment from (2,0,0), C_2, the line segment from (0,2,0) to (0,0,2), and C_3, then line segment from (0,0,2), to (2,0,0). By property (iii),

$$\int_C xy + z^2 \; ds = \int_{C_1} xy + z^2 \; ds + \int_{C_2} xy + z^2 \; ds + \int_{C_3} xy + z^2 \; ds.$$

The parametrizations for these curves, respectively, are:

$$\begin{aligned}\mathbf{r}_1(t) &= (2-2t, 2t, 0), \;\; 0 \le t \le 1 \\ \mathbf{r}_2(t) &= (0, 2-2t, 2t), \;\; 0 \le t \le 1 \\ \mathbf{r}_3(t) &= (2t, 0, 2-2t), \;\; 0 \le t \le 1.\end{aligned}$$

It is easy to check that $||\mathbf{r}_i(t)|| = 8$, for all i. Thus,

$$\begin{aligned}\int_{C_1} xy + z^2 \; ds &= \int_0^1 (2-2t)2t + 0^2 \; \sqrt{8} \; dt \\ &= \frac{2\sqrt{8}}{3} \\ \int_{C_2} xy + z^2 \; ds &= \int_0^1 0(2-2t) + (2t)^2 \; \sqrt{8} \; dt \\ &= \frac{4\sqrt{8}}{3} \\ \int_{C_3} xy + z^2 \; ds &= \int_0^1 2t \cdot 0 + (2-2t)^2 \; \sqrt{8} \; dt \\ &= \frac{4\sqrt{8}}{3}.\end{aligned}$$

Therefore,

$$\int_C xy + z^2 \; ds = \frac{2\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} + \frac{4\sqrt{8}}{3} = \frac{10\sqrt{8}}{3}.$$

Example. Let C be the upper half of the unit circle in the xy-plane, centered at the origin. We verified that $\int_C f(x, y, z)\ ds$ is independent of the parametrization for $f(x, y) = ye^x$ and the parametrizations: $\mathbf{r}(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi$ and $\mathbf{s}(t) = (\cos(\pi - 2t), \sin(\pi - 2t))$, $0 \le t \le \frac{\pi}{2}$. For $\mathbf{r}(t)$ we have

$$||\mathbf{r}'(t)|| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1,$$

so that

$$\begin{aligned}\int_C ye^x\ ds &= \int_0^{\pi} \sin(t)e^{cos(t)}\ dt \\ &= -e^{\cos(t)}\Big|_{t=0}^{t=\pi} \\ &= -e^{-1} + e.\end{aligned}$$

For the parametrization $\mathbf{s}(t)$ we have

$$||\mathbf{s}'(t)|| = \sqrt{(2\cos(\pi - 2t))^2 + (-2\sin(\pi - 2t)^2} = 2,$$

so that,

$$\int_C ye^x\ ds = \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi - 2t)}\ 2\ dt.$$

Using u-substitution with $u = \cos(\pi - 2t)$, we have

$$\begin{aligned}\int_C ye^x\ ds &= \int_0^{\frac{\pi}{2}} \sin(\pi - 2t)e^{\cos(\pi - 2t)}\ 2\ dt. \\ &= \int_{-1}^{1} e^u\ du \\ &= e - e^{-1},\end{aligned}$$

which agrees with the calculation above.

We then stated the following applications of a line integral.

Applications.

(i) $\frac{1}{\text{length}(C)} \int_c f(x, y, z)\ ds$ give the average value of $f(x, y, z)$ over C.

(ii) If $C \subseteq \mathbb{R}^2$ and $f(x, y) \ge 0$, for $(x, y) \in C$, then $\int_C f(x, y)\ ds$ represents the area under the surface $z = f(x, y)$ above the curve C.

(iii) If C represents a wire and $f(x, y, z)$ is the density of the wire at the point (x, y, z), then the *total mass* of the wire is $M = \int_C f(x, y, z)\ ds$.

(iv) The *center of mass* of the wire is the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\int_C xf(x, y, z)\ ds}{M}, \quad \overline{y} = \frac{\int_C yf(x, y, z)\ ds}{M}, \quad \overline{z} = \frac{\int_C zf(x, y, z)\ ds}{M}.$$

We ended our initial discussion of line integrals by showing that if we are given two 1-1 parameterizations of the curve C, $\mathbf{r}(t)$, $a \le t \le b$ and $\mathbf{s}(t)$, $c \le t \le d$, with continuous derivatives, then

$$\int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)||dt = \int_c^d f(\mathbf{s}(t))||\mathbf{s}'(t)||dt.$$

This was accomplished by viewing $\mathbf{s}(t)$ as a *re-parameterization* of $\mathbf{r}(t)$, i.e., by writing $\mathbf{s}(t) = \mathbf{r}(\theta(t))$, for $\theta : [c, d] \to [a, b]$, where θ is 1-1, with a continuous derivative, and $\theta(c) = a$ and $\theta(d) = b$.

Friday, November 7. We then began our discussion of surface integrals whose integrands are scalar functions. We follow the same process used in all previous forms of integration. To integrate $f(x, y, z)$ over the surface S

Step 1. Subdivide S into finitely many smaller surfaces S_i of the same area ΔS. We are using ΔS for a small element of surface area.

Step 2. Choose a point (x_i, y_i, z_i) from the component S_i.

Step 3. Multiply $f(x_i, y_i, z_i)$ by the size of each S_i to get $f(x_i, y_i, z_i)\ \Delta S$.

Step 4. Add the products in Step 3 to get the Riemann sum: $\Sigma_i f(x_i, y_i, z_i)\Delta S$.

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\int\int_S f(x, y, z)\ dS,$$

the surface integral of $f(x, y,)$ over S with respect to surface area. Note that we write a double integral, since our domain of integration is two-dimensional.

Following an analogy with curves, to calculate $\int\int_S f(x, y, z)\ dS$, we will need:

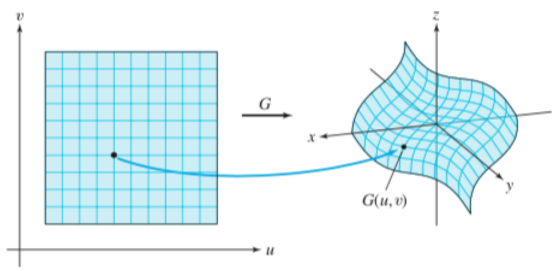
(i) A way to describe or *parametrize* a surface as a function of two variables.

(ii) A way to calculate surface area.

Definition. Given a surface $S \subseteq \mathbb{R}^3$, a parametrization of S will be a function

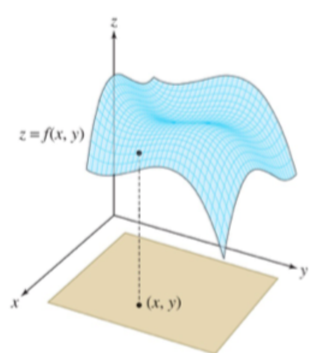
$$G(u, v) = (x(u, v), y(u, v), z(u, x)),$$

such that $S = G(D)$ for some domain D in the uv-plane.



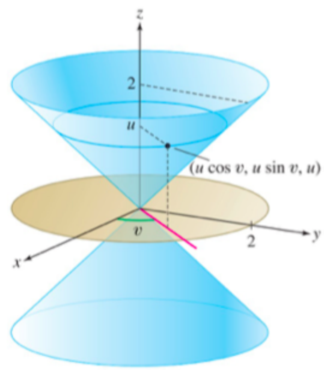
As usual, we assume that all first order partials exists and are continuous, at least on the interior of D.

Example 1. The easiest surface to parametrize is a surface that is the graph of $z = f(x, y)$. Why: Because it is already defined by two parameters!



Here we can write $G(u, v) = (u, v, f(u, v))$ or $G(x, y) = (x, y, f(x, y))$. For example, if S is that portion of the paraboloid $z = x^2 + y^2$ lying over $D : 0 \leq x^2 + y^2 \leq 9$, then $G(u, v) = (u, v, u^2 + v^2)$, with $0 \leq u^2 + v^2 \leq 9$ is a parametrization that takes the disk of radius 3 in the uv-plane to S.

Example 2. Consider the cone given by $z^2 = x^2 + y^2$, with $0 \leq x^2 + y^2 \leq 4$.



Though the top half of the cone can be expressed as $z = \sqrt{x^2 + y^2}$ and we could parametrize it by $G(u, v) = (u, v, \sqrt{u^2 + v^2})$ with $0 \le x^2 + y^2 \le 4$, the whole surface cannot be expressed as the graph of a function of x and y.

Better: $G(u, v) = (u\cos(v), u\sin(v)u)$, with $-2 \le u \le 2$ and $0 \le v \le 2\pi$. Note

$$u^2 = (u\cos(v))^2 + (u\sin(v))^2,$$

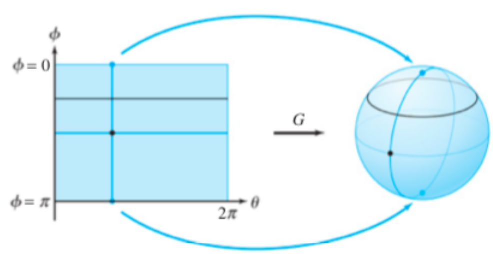
so the points $G(u, v)$ lie on the cone. More over, if we hold u fixed at u_0 and let v vary between 0 and 2π, this vertical line segment in the uv-plane is taken by G to the circle $G(u_0, v) = (u_0\cos(v), u_0\sin(v), u_0)$, i.e., the circle of radius u_0 centered at $(0, 0, u_0)$. As u_0 varies between 2 and -2, these circles sweep out the cone.

Spherical and cylindrical coordinates tell us how to parametrize spheres and cylinders.

Example 3. For S the sphere of radius R centered at the origin we take:

$$G(\phi, \theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \le \phi \le \pi, 0 \le \theta < 2\pi$.



As we have seen in our earlier discussion concerning spherical coordinates, this transformation take the vertical line segment $\phi = \phi_0$, with $0 \le \theta \le 2\pi$ in the $\phi\theta$-plane to the circle $(R\sin(\phi_0)\cos(\theta), R\sin(\phi_0)\cos(\theta), R\cos(\phi_0))$ of radius $R\sin(\phi_0)$ centered at $(0, 0, R\cos(\phi_0))$. As ϕ_0 varies from 0 to π these circles sweep our the sphere.

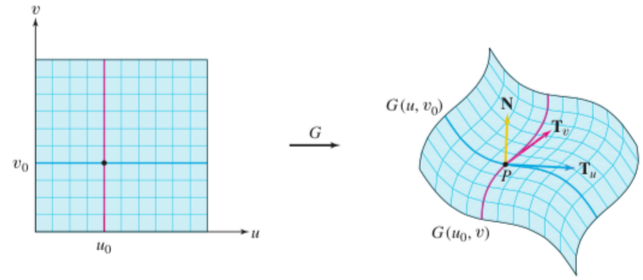
Now that we have a description of a surface in terms of parameters u, v, we can use this parameterization to find the plane tangent to the surface at $P = (x_0, y_0, z_0) = G(u_0, v_0)$. Since the equation of any plane is determined by a point on the plane and a vector normal to the plane, to find the tangent plane to the surface S at a point P, we find two tangent vectors, and take their cross product to find a normal vector.

Suppose $P = G(u_0, v_0)$. How do we get tangent vectors to S at P? Hold v fixed at v_0 and let u vary. Then $G(u, v_0)$ gives a curve C_1 that moves in the u-direction on S passing through P.
Thus, $\mathbf{T}_u(u_0, v_0) = \frac{\partial G}{\partial u}(u_0, v_0)$ is a vector tangent to C_1, and hence S, at P.

$$\mathbf{T}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)i + \frac{\partial y}{\partial u}(u_0, v_0)j + \frac{\partial z}{\partial u}(u_0, v_0)k.$$

We get a second tangent vector to S by taking a tangent to the curve obtained by holding u fixed at u_0.

$$\mathbf{T}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)i + \frac{\partial y}{\partial v}(u_0, v_0)j + \frac{\partial z}{\partial v}(u_0, v_0)k.$$



The normal vector to S at P is

$$\mathbf{N} = \mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0),$$
$$= \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u}(u_0, v_0) & \frac{\partial y}{\partial u}(u_0, v_0) & \frac{\partial z}{\partial u}(u_0, v_0) \\ \frac{\partial x}{\partial v}(u_0, v_0) & \frac{\partial y}{\partial v}(u_0, v_0) & \frac{\partial z}{\partial v}(u_0, v_0) \end{vmatrix}.$$

Thus, the equation of the plane tangent to S at the point $P = G(u_0, v_0) = (x_0, y_0, z_0)$ is given by

$$\mathbf{T}_u(u_0, v_0) \times \mathbf{T}_v(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Monday, November 10 After a brief reminder about parameterized surfaces, we considered

Formula for surface area. If the bounded surface S is given by $G(u, v)$, with $G(u, v)$ 1-1 on the interior of D and $G(D) = S$, for D in the uv-plane, then:

$$\text{surface area}(S) = \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dA,$$

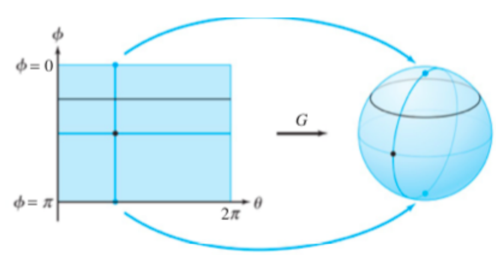
where the double integral is a standard double integral in the uv-plane.

Example 5. Find the surface area of a sphere S of radius R.

Solution. We use

$$G(\phi, \theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$, which defines D.



$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$\mathbf{T}_\phi \times \mathbf{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix}$$
$$= -R^2\sin(\phi)\{\sin(\phi)\cos(\theta)\vec{i} + \sin(\phi)\sin(\theta)\vec{j} + \cos(\phi)\vec{k}\}$$

We have:

$$||\mathbf{T}_\phi \times \mathbf{T}_\theta|| = R^2\sin(\phi),$$

since $(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$ lies on the sphere of radius 1. Note that $R^2\sin(\phi)$ is non-negative since $0 \leq \phi \leq \pi$. Thus:

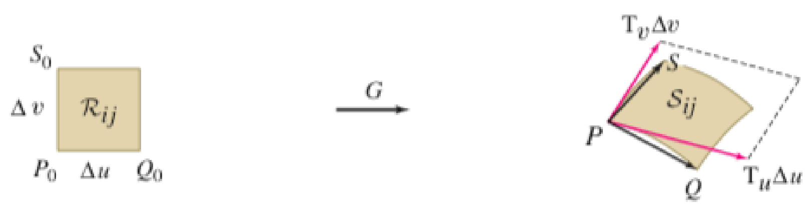
$$\begin{aligned} \text{surface area}(S) &= \int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA \\ &= \int_0^{2\pi}\int_0^{\pi} R^2\sin(\phi)\ d\phi d\theta \\ &= 2\pi R^2 \int_0^{\pi} \sin(\phi)\ d\phi \\ &= 2\pi R^2(-\cos(\phi))\Big|_0^{\pi} \\ &= 4\pi R^2. \end{aligned}$$

Question: Do you see a connection between the formula for the volume of a sphere of radius R and its surface area?

Answer: The surface area is the derivative of the volume.

In order to understand our formulas for calculating surface area and surface integrals, we need to understand, where does the formula for surface area come from? For this we will use the fact from Calculus 2 that if $\mathbf{v}_1 = ai+bj+ck$ and $\mathbf{v}_2 = di+cj+ek$, then the area of the parallelogram spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $||\mathbf{v}_1 \times \mathbf{v}_2||$.

We first subdivide S into small portions, S_i with surface area ΔS. We approximate the small portions of surface area ΔS with small approximating tangent parallelograms. We start with the parametrization $G(u, v)$ of S. We use a tangent parallelogram to estimate the surface of the curved parallelogram on the surface.



Note that the vector from P to Q is $G(u_0 + \Delta u, v_0) - G(u_0, v_0)$. This vector is approximated by the tangent vector in red, $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$. Similarly, the other tangent vector in red $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$ approximates the vector $G(u_0, v_0 + \Delta v) - G(u_0, v_0)$. Thus, our small approximating tangent parallelogram is spanned by the vectors $\Delta u \cdot \mathbf{T}_u(u_0, v_0)$ and $\Delta v \cdot \mathbf{T}_v(u_0, v_0)$. The area of the approximating rectangle is

$$||(\Delta u \cdot \mathbf{T}_u) \times (\Delta v \cdot \mathbf{T}_v)|| = ||\mathbf{T}_\mathbf{u}(u_0, v_0) \times \mathbf{T}_v(u_0, v_0)||\Delta u \Delta v.$$

Adding these areas for each component of the subdivision and taking the limit as $\Delta u, \Delta v \to 0$, gives a total surface area of $\int\int_D ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

By retracing our steps in defining a surface integral, using a parameterization leads to a formula for calculating $\int\int_S f(x, y, z)\ dS$.

Step 1. Subdivide S into finitely many smaller surfaces S_i of area $\Delta S \approx ||\mathbf{T}_u \times \mathbf{T}_v||\ dA$.

Step 2. Choose a point $G(u_i, v_i)$ from the component S_i.

Step 3. Multiply $f(G(u_i, v_i))$ the area of S_i to get

$$f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 4. Add the products in Step 3 to get the Riemann sum :

$$\Sigma_i f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_i)||\ dA.$$

Step 5. Take the limit of the Riemann sums as $\Delta S \to 0$, to get:

$$\begin{aligned}\int\int_S f(x,y,z)\ dS &= \int\int_D f(G(u_i, v_i)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dA \\ &= \int\int_D f((x(u,v), y(u,v), z(u,v)) \cdot ||(\mathbf{T}_u \times \mathbf{T}_v||\ dudv.\end{aligned}$$

We then calculated the following examples.

Example. Calculate $\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS$ for S the paraboloid $z = x^2 + y^2$, $0 \le z \le 4$.

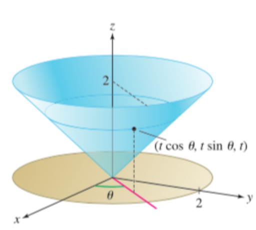
Solution. We take $G(u,v) = (u, v, u^2 + v^2)$, with $0 \le u^2 + v^2 \le 4$. We need to calculate $||\mathbf{T}_u \times \mathbf{T}_v||$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2ui - 2vj + k.$$

Thus $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-2u)^2 + (-2v)^2 + 1^2} = \sqrt{4u^2 + 4v^2 + 1}$. Using this in the formula for surface integrals, we have

$$\begin{aligned}\int\int_S \frac{1}{1+4(x^2+y^2)}\ dS &= \int\int_D \frac{1}{1+4(u^2+v^2)} \cdot \sqrt{1+4u^2+4v^2}\ dA \\ &= \int\int_D (1+4u^2+4v^2)^{-\frac{1}{2}}\ dudv \\ &= \int_0^{2\pi}\int_0^2 (1+4(r\cos(\theta))^2 + 4(r\sin(\theta))^2)^{-\frac{1}{2}}\ rdrd\theta \\ &= 2\pi \int_0^2 (1+4r^2)^{-\frac{1}{2}}\ rdr \\ &= 2\pi \cdot \frac{1}{4}(1+4r^2)^{\frac{1}{2}}\Big|_0^2 \\ &= \frac{\pi}{2}\{\sqrt{17} - 1\}.\end{aligned}$$

Example. Calculate $\int\int_S x^2 z\ dS$, where S is that portion of the cone $S : z^2 = x^2 + y^2$ lying above the disk $D : 0 \le x^2 + y^2 \le 4$.



Solution. From last lecture, $G(u,v) = (u\cos(v), u\sin(v), u)$, with $0 \le u \le 2$, $0 \le v \le 2\pi$, is a parametrization of S.

$\mathbf{T}_u = (\cos(v), \sin(v), 1)$ and $\mathbf{T}_v = (-u\sin(v), u\cos(v), 0)$.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$
$$= (-u\cos(v), -u\sin(v), u)$$

$||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{(-u\cos(v))^2 + (-u\cos(v))^2 + u^2} = \sqrt{2u^2} = \sqrt{2}|u|$.

$$\int\int_S x^2 z \; dS = \int\int_D (ucos(v))^2 u \cdot \sqrt{2}|u| \; dudv$$
$$= \sqrt{2}\int_0^{2\pi}\int_0^2 u^4\cos^2(v) \; dudv,$$

since u is non-negative on D.

$$= \sqrt{2}\cdot\frac{32}{5}\int_0^{2\pi}\cos^2(v) \; dv$$
$$= \frac{32\sqrt{2}}{5}\int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2v) \; dv$$
$$= \frac{32\sqrt{2}}{5}(\frac{v}{2} + \frac{1}{4}\sin(2v))\Big|_0^{2\pi}$$
$$= \frac{32\sqrt{2}}{5}\pi$$

We then recorded the following familiar looking properties for surface integrals of continuous functions defined on a surface S:

(i) $\int\int_S f + g \; dS = \int\int_S f \; dS + \int\int_S g \; dS$.
(ii) $\int\int_S \lambda f \; dS = \lambda\int\int_S f \; dS$, for $\lambda \in \mathbb{R}$.
(iii) If $S = S_1 \cup S_2$, then $\int\int_S f \; dS = \int\int_{S_1} f]dS + \int\int_{S_2} f \; dS$, as long as S_1 and S_2 only intersect along their boundaries.
(iv) Surface area$(S) = \int\int_S \; dS$.

Important comments about calculating surface area and surface integrals.

1. The quantity $||\mathbf{T}_u \times \mathbf{T}_v||$ should always be non-negative (and almost never zero). If you you calculate this expression and it involves variables from the parametrization, this function should be positive on the domain of integration. For example, if you get $||\mathbf{T}_u \times \mathbf{T}_v|| = 2uv$, then you have to make sure that the product uv is positive on the domain of integration. If not, you may have dropped an absolute value somewhere.

2. Aside from having component functions with continuous first order partial derivatives, $G(u, v)$ should be 1-1 on the interior of the domain D. Otherwise, one may be double counting portions of the surface area.

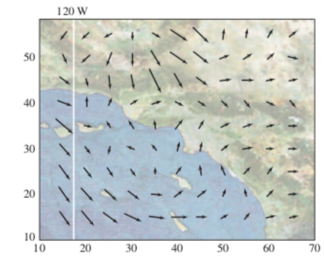
3. For calculating surface area and surface integrals of scalar functions, the orientation of the normal vector does not matter, because these quantities use the length of the normal vector. However, it will matter when we integrate vector valued functions over a surface. After all, we care whether or not a fluid flows into or out of a chamber. Or as Hagrid might say: Better out than in.

Wednesday, November 12. We began a discussion of integrating vector fields over curves and surfaces.

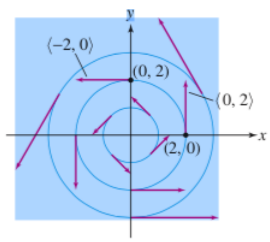
Definition. A vector field is a vector valued function

$$\mathbf{F}(x, y, z) = F_1(x, y, z)i + F_2(x, y, z,)j + F_3(x, y, z)k,$$

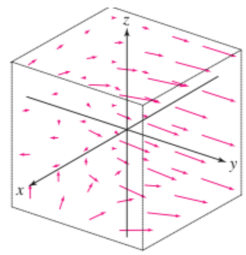
that assigns a vector to each point in a subset $S \subseteq \mathbb{R}^3$. Here's a picture of wind speed and wind direction as a vector field.



We will make our usual assumptions that the first order partial derivatives of all component functions exist and are continuous in suitable regions contained in $\mathbb{R}^3$. Some other pictures of vector fields:



$\mathbf{F} = \langle -y, x \rangle$



(A) $\mathbf{F} = \langle x \sin z, y^2, x/(z^2+1) \rangle$

Tangent vector fields and normals vector fields play a central role in what we want to do next.

Case 1. Let C be a smooth curve. Then at each point $P = (x, y, z)$ of the curve, we can assign a tangent vector $\mathbf{F}(x, y, z)$ to C at the point P. This gives a vector field along the curve C. Note that if $\mathbf{r}(t)$ is a parametrization of C, then $\mathbf{r}'(t)$ is a tangent vector at each point along the curve. Recall that $\mathbf{r}'(t)$ points in the direction of a point traveling along the curve and the length of $\mathbf{r}'(t)$ gives the speed of the point at time t.

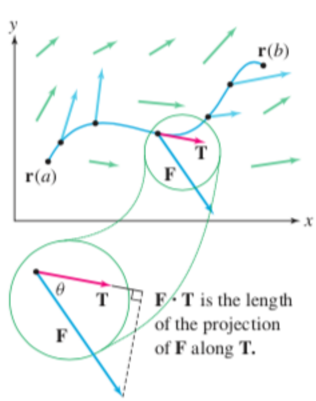
If we want to just keep track of the direction of a point moving along the curve, we can consider the unit tangent vector $\mathbf{T}(x, y, z)$ along the curve. If $\mathbf{r}(t)$ is a parametrization of C and $\mathbf{r}'(t) \neq 0$, then

$$\mathbf{T}(x, y, z) = \frac{1}{||\mathbf{r}'(t)||} \cdot \mathbf{r}'(t).$$

How do we use the unit tangent vectors? If we have a vector field $\mathbf{F}$ pushing a point along a curve, the dot product $\mathbf{F}(P) \cdot \mathbf{T}(P)$ gives us the component of $\mathbf{F}$ in the direction of the curve since:

$$\mathbf{F}(P) \cdot \mathbf{T}(P) = ||\mathbf{F}(P)|| \cdot ||\mathbf{T}(P)|| \cos(\theta) = ||\mathbf{F}(P)|| \cos(\theta),$$

where θ is the angle between $\mathbf{F}(P)$ and $\mathbf{T}(P)$.



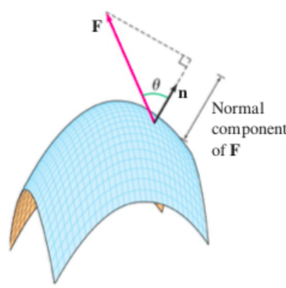
Informally, we can think of $\mathbf{F} \cdot \mathbf{T}$ as "how much of $\mathbf{F}$ acts in the direction of $\mathbf{T}$". We will see that when we want to integrate a vector field $\mathbf{F}$ along the curve C, we will really be integrating the scalar function $\mathbf{F} \cdot \mathbf{T}$ along the C.

Case 2. If we have a smooth surface S, then the normal vector $\mathbf{N}$ at each point of S defines a vector field on S.

Let S be a smooth surface in $\mathbb{R}^3$. We obtain a vector field $\mathbf{F}(x, y, z)$ by assigning to each point P on the surface a vector that is normal to S at P. At each point there are two normal vectors, each the negative of the other. The parametrization of the surface may not always yield the desired vector. If $G(u, v)$ is a parametrization of S, then $\mathbf{N}(u, v) = \mathbf{T}_u \times \mathbf{T}_v$ gives a normal vector and

$$\mathbf{n}(u, v) = \frac{1}{||\mathbf{T}_u(u, v) \times \mathbf{T}(u, v)_v||} \cdot \mathbf{N}(\mathbf{u}, \mathbf{v}),$$

gives a unit normal to the surface. If we have a vector field $\mathbf{F}$ passing through a surface - think of a fluid passing through a membrane - then $\mathbf{F}(P) \cdot \mathbf{n}(P)$ gives the component of $\mathbf{F}(P)$ in the direction of $\mathbf{n}(P)$. In fluid mechanics, one would call this the flux of the fluid across the boundary S at the point P.



We can now define line and surface integrals of vector fields.

Definition. Given a curve $C : \mathbf{r}(t)$ and a vector field $\mathbf{F}$, the line integral of $\mathbf{F}$ along C is the line integral of the scalar function $\mathbf{F} \cdot \mathbf{T}$, and is denoted $\int_C \mathbf{F} \cdot d\mathbf{r}$. Here $\mathbf{T}$ is the unit tangent along the curve pointing in the direction we travel along the curve. In other words,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\ ds,$$

where the quantity on the right is the line integral along C of the scalar function $\mathbf{F} \cdot \mathbf{T}$. In terms of calculating the line integral, given the parametrization $\mathbf{r}(t)$ we have

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (\mathbf{F} \cdot \mathbf{T})\ ds \\ &= \int_a^b \{\mathbf{F}(\mathbf{r}(t)) \cdot (\frac{1}{||\mathbf{r}'(t)||}\mathbf{r}'(t))\}\ ||\mathbf{r}'(t)||dt \\ &= \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\ dt,\end{aligned}$$

since $\frac{1}{||\mathbf{r}'(t)||}$ cancels with $||\mathbf{r}'(t)||$. Thus, we do not have to calculate $||\mathbf{r}'(t)||$ even though it is implicit in the definition of $\int_C \mathbf{F} \cdot d\mathbf{r}$. Note that to calculate $\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}(x(t), y(t), z(t))$, everywhere x, y, or z appear in the formula for $\mathbf{F}$ we replace these by $x(t), y(t), z(t)$ given in the parametrization of C.

Example. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, for C given by $\mathbf{r}(t) = (t + 1, e^t, t^2),\ \ 0 \le t \le 2$ and $\mathbf{F} = zi + y^2 j + xk$.

Solution. $\mathbf{F}(\mathbf{r}(t)) = t^2 i + (e^t)^2 j + (t + 1)k$ and $\mathbf{r}'(t) = i + e^t j + 2tk$. Thus,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2 + e^{3t} + 2t(t + 1) = e^{3t} + 2t + 3t^2.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 e^{3t} + 2t + 3t^2 \ dt = \frac{1}{3}(e^6 + 35).$$

Definition. Given an surface S parametrized by $G(u,v) = (x(u,v), y(u,v), z(u,v))$, and a vector field $\mathbf{F}$, the surface integral of $\mathbf{F}$ over S is the surface integral of the scalar function $\mathbf{F} \cdot \mathbf{n}$, and is denoted $\int\int_S \mathbf{F} \cdot d\mathbf{S}$. Here, $\mathbf{n}$ is the unit normal on the surface determined by the parametrization, i.e., $\mathbf{n} = \frac{1}{||\mathbf{T}_u \times \mathbf{T}_v||} \mathbf{T}_u \times \mathbf{T}_v$

In other words, by definition,

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S (\mathbf{F} \cdot \mathbf{n}) \ dS.$$

We calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ using a parametrization. If

$$G(u,v) = (x(u,v), y(u,v), z(u,v)), \text{ for } (u,v) \in D$$

is a parametrization of S, the surface integral of $\mathbf{F}$ over S is given by

$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_S \mathbf{F} \cdot \mathbf{n} \ dS \\ &= \int\int_D \mathbf{F}(G(u,v)) \cdot \{\frac{1}{||\mathbf{T}_u \times \mathbf{T}_v||} \mathbf{T}_u \times \mathbf{T}_v\} \ ||\mathbf{T}_u \times \mathbf{T}_v|| \ dA \\ &= \int\int_D \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v \ dA. \end{aligned}$$

Thus, in a similar vein to the line integral above, in calculating $\int\int_S \mathbf{F} \cdot d\mathbf{S}$, we do not need to calculate $||\mathbf{T}_u \times \mathbf{T}_v||$, even though this quantity is implicit in the definition of $\int\int_S \mathbf{F} \cdot d\mathbf{S}$. Note also, that ultimately, the calculation of $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ reduces to the calculation of a double integral of a function of u and v over the flat region D in the uv-plane. Moreover, $\mathbf{F}(G(u,v))$ is obtained by replacing each occurrence of x, y, z in the definition of $\mathbf{F}$ by $x(u,v), y(u,v), z(u,v)$ respectively.

Example. Calculate $\int\int \mathbf{F} \cdot d\mathbf{S}$ for S the upper hemisphere of the sphere of radius R centered at the origin, with the standard parametrization, and $\mathbf{F} = zi + xj + k$.

Solution. We have

$$G(\phi, \theta) = (R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)),$$

with $0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta < 2\pi$, which defines D.

$$\mathbf{T}_\phi = (R\cos(\phi)\cos(\theta), R\cos(\phi)\sin(\theta), -R\sin(\phi)) \quad \text{and} \quad \mathbf{T}_\theta = (-R\sin(\phi)\sin(\theta), R\sin(\phi)\cos(\theta), 0).$$

Thus,

$$\begin{aligned} \mathbf{T}_\phi \times \mathbf{T}_\theta &= \begin{vmatrix} i & j & k \\ R\cos(\phi)\cos(\theta) & R\cos(\phi)\sin(\theta) & -R\sin(\phi) \\ -R\sin(\phi)\sin(\theta) & R\sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\ &= R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \end{aligned}$$

On the other hand,

$$\mathbf{F}(G(u,v)) = (R\cos(\phi), R\sin(\phi)\cos(\theta), 1),$$

Thus,

$$\begin{aligned} \mathbf{F}(G(u,v) \cdot \mathbf{T}_u \times \mathbf{T}_v &= (R\cos(\phi), R\sin(\phi)\cos(\theta), 1) \cdot R^2\sin(\phi)(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)) \\ &= R^2\sin(\phi)\{R\cos(\phi)\sin(\phi)\cos(\theta) + R\sin(\phi)\cos(\theta)\sin(\phi)\sin(\theta) + \cos(\phi)\} \\ &= R^3\sin(\phi)\{\cos(\phi)\sin(\phi)\cos(\theta) + \sin^2(\phi)\cos(\theta)\sin(\theta) + \cos(\phi)\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= R^3\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2\phi\cos(\theta)+\sin^3(\phi)\cos(\theta)\sin(\theta)+\cos(\phi)\sin(\phi)\ d\phi\ d\theta\\
&= R^3\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos(\phi)\sin^2\phi\cos(\theta)+\sin^3(\phi)\cos(\theta)\sin(\theta)+\cos(\phi)\sin(\phi)\ d\theta\ d\phi\\
&= R^3\int_0^{\frac{\pi}{2}} \cos(\phi)\sin^2(\phi)(-\sin(\theta))_{\theta=0}^{\theta=2\pi}+\sin^3(\phi)(\frac{1}{2}\sin^2(\theta))_{\theta=0}^{\theta=2\pi}+\cos(\phi)\sin(\phi)\theta|_{\theta=0}^{\theta=2\pi}\ d\phi\\
&= 2\pi R^3\int_0^{\frac{\pi}{2}}\cos(\phi)\sin(\phi)\ d\phi\\
&= \pi R^3\sin^2(\phi)\Big|_0^{\phi=\frac{\pi}{2}}\\
&= \pi R^3.
\end{aligned}$$

Friday, November 14. We began class by recalling and discussing the formulas presented in the last lecture for integrating vectors fields along a curve or surface.

We then looked at the following interesting example.

Example. For $\mathbf{F} = (3y+1)\vec{i} + 3x\vec{j}$, calculate $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}$ nd $\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, for the curves $C_1 : (\cos(t), \sin(t))$, $0 \le t \le \frac{\pi}{2}$ and $C_2 : (1-t, t)$, $0 \le t \le 1$ connecting the points $P = (1, 0)$ and $Q = (0, 1)$.

Solution. For C_1, we have

$$\begin{aligned}
\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}}(3\sin(t)+1, 3\cos(t))\cdot(-\sin(t),\cos(t)\ dt\\
&= \int_0^{\frac{\pi}{2}}(-3\sin^2(t)-\sin(t)+3\cos^2(t))\ dt\\
&= \int_0^{\frac{\pi}{2}}3(\cos^2(t)-\sin^2(t))-\sin(t)\ dt\\
&= \int_0^{\frac{\pi}{2}}3\cos(2t)-\sin(t)\ dt\\
&= (\frac{3}{2}\sin(2t)+\cos(t))\Big|_0^{\frac{\pi}{2}}\\
&= -1.
\end{aligned}$$

For C_2 we have

$$\begin{aligned}
\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^1(3t-1, 3(1-t))\cdot(-1,1)\ dt\\
&= \int_0^1 -6t+2\ dt\\
&= (-3t^2+2t)\Big|_0^1\\
&= -1.
\end{aligned}$$

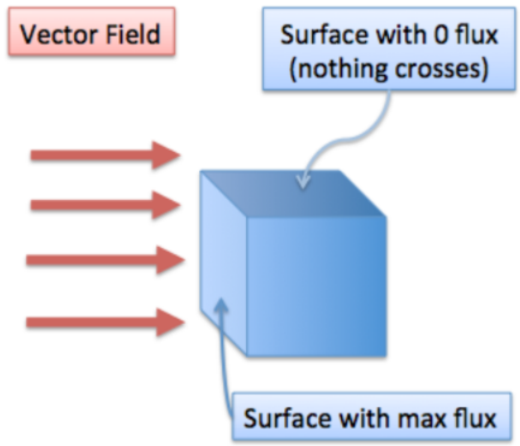
We noted that this example illustrated the concept of *path independence* enjoyed by certain vector fields $\mathbf{F}$, the so-called *conservative* vector fields. We noted that in this case $\mathbf{F} = \nabla f$, for $f(x, y) = 3xy + x$, and recorded, but did not discuss at length, the formula $\int_C \nabla f\cdot d\mathbf{r} = f(Q) - f(P)$, for the smooth curve C starting at the point P and ending at the point Q.

We then began a discussion of the concept of the *flux* of vector field.

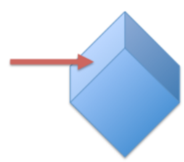
If we imagine that the vector field **F** represents a fluid passing through a membrane (e.g., blood through capillaries or an electromagnetic field passing though a shield) then, by definition, the flux is the quantity that passes through the membrane or shield. We want to see that the flux can be calculated as a surface integral.

Case 1. First consider a constant vector field **F** passing through a rectangle R perpendicular to the vectors in **F**. How much of **F** passes through R? Answer:

$$||\mathbf{F}|| \cdot \text{area}(R) = (\mathbf{F} \cdot \frac{1}{||\mathbf{F}||}\mathbf{F}) \cdot \text{area}(R) = (\mathbf{F} \cdot \mathbf{n}) \cdot \text{area}(R).$$



Note when **F** is parallel to R, the flux is zero. In this case $\mathbf{F} \cdot \mathbf{n} = 0 =$ flux. Case 2. **F** is constant, passing through R, neither perpendicular or parallel to R.

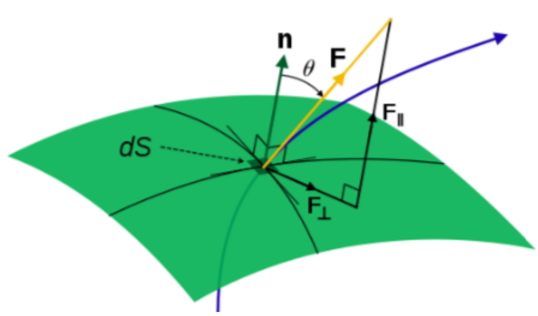


The amount of **F** flowing though R is $(\mathbf{F} \cdot \mathbf{n}) \cdot \text{area}(R)$.

General case. **F** is not constant, and the surface S is not a rectangle. Subdivide S into very small sections S_i of size dS. For $P_i \in S_i$, **F** on S_i is approximately $\mathbf{F}(P_i)$. The flux across each S_i is approximately: $\mathbf{F}(P_i) \cdot \mathbf{n}(P_i)\ dS$. Adding over the surface gives a Riemann sum:

$$\sum_i \mathbf{F}(P_I) \cdot \mathbf{n}(P_i)\ dS.$$

Taking the limit as $dS \to 0$ gives $\int\int_S \mathbf{F} \cdot \mathbf{n}\ dS = \int\int_S \mathbf{F} \cdot d\mathbf{S}$.



Therefore, we get the Flux of **F** passing through S equals $\int\int_S \mathbf{F} \cdot \mathbf{n}\ dS$.

We then did two examples of calculating flux.

Example. Let S denote the closed cylinder of radius a and height h whose base is the disk of radius a centered at the origin in $\mathbb{R}^2$ and $\mathbf{F} = y\vec{i} + z\vec{j} + x\vec{k}$. We saw that $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0$, by integrating separately over the top, bottom, and side of the closed cylinder. In fact, each of these three integral were zero.

Example. Suppose S is the sphere of radius one centered at the origin in $\mathbb{R}^3$ and $\mathbf{F} = x\vec{i}+y\vec{j}+z\vec{k}$. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$. We worked this by noting that $bfn = x\vec{i} + y\vec{j} + z\vec{k}$ on S, so that $\mathbf{F} \cdot \mathbf{n} = x^2 + y^2 + z^2$, which is 1 everywhere on S. Thus, $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S 1\ dS = 4\pi$, the surface area of S.

Monday, November 17. We began a discussion of the divergence of a vector field and the Divergence Theorem.

Suppose we wish to interpret how the flow of a vector field $\mathbf{F}$ behaves near a point P. For example, a point source of a radiating electromagnetic field or a sink attracting a fluid in motion. Ideally, this should tell us whether or not, on the whole, the field $\mathbf{F}$ is flowing towards or away from the point P, and with what magnitude. A first attempt might be to consider a small closed sphere S_ϵ of radius ϵ about P, and then take the limit of the flux of $\mathbf{F}$ through S_ϵ as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \int\int_{S_\epsilon} \mathbf{F} \cdot d\mathbf{S} = \lim_{\epsilon\to 0} \int\int_{S_\epsilon} \mathbf{F} \cdot \mathbf{n}\ dS.$$

The problem with this is that since the surface area of S_ϵ tends to zero as $\epsilon \to 0$, this limit will always be zero. If we think of S_ϵ as being full of a liquid, and we empty this liquid as $\mathbf{F}$ moves from P, this all passes through the boundary of S_ϵ as $\epsilon \to 0$. In other words, what we want is the flux per unit volume of $\mathbf{F}$ over S_ϵ, as $\epsilon \to 0$, which would be

$$\lim_{\epsilon\to 0} \frac{1}{\text{vol}(S_\epsilon)} \int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

This limit is called the *Divergence* of $\mathbf{F}$ at the point P. Letting P vary, we have a new **scalar** function, called the divergence of $\mathbf{F}$, which we denote by div $\mathbf{F}$.

Important facts about the Divergence of $\mathbf{F}$.

(i) The divergence does not depend upon the coordinate system used.

(ii) The divergence of $\mathbf{F}$ at P can be calculated as

$$\lim_{\text{vol}(S)\to 0} \frac{1}{\text{vol}(S)} \int\int_S \mathbf{F} \cdot d\mathbf{S}$$

where the limit is taken over any sequence of (sufficiently smooth) closed surfaces containing P, whose volumes tend to 0.

(iii) div $\mathbf{F}$ is a scalar function of x, y, z.

(iv) $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div}\ \mathbf{F} + \text{div}\ \mathbf{G}$.

(v) $\lambda \cdot \text{div}\ \mathbf{F} = \text{div}(\lambda\mathbf{F})$, for any $\lambda \in \mathbb{R}$.

Example. Let $\mathbf{F} = x^3\vec{i} + 4y\vec{j} + 2z^2\vec{k}$. Calculate div $\mathbf{F}(x_0, y_0, z_0)$, using cubes centered at $P = (x_0, y_0, z_0)$.

Solution. Let C_s denote the cube centered at P having sides of length s. Then,

$$\text{div}\ \mathbf{F}(P) = \lim_{s\to 0} \frac{1}{s^3} \int\int_{C_s} \mathbf{F} \cdot d\mathbf{S} = \lim_{s\to 0} \frac{1}{s^3} \int\int_{C_s} \mathbf{F} \cdot \mathbf{n}\ dS.$$

We first evaluate the surface integral of $\mathbf{F}$ over the front and back faces of C_s. For the front face S_1, we note that $\mathbf{n} = \vec{i}$ and that the x coordinate of every point on the front fact is $x_0 + \frac{s}{2}$. Since $\mathbf{F} \cdot \mathbf{n} = x^3$, it follows that on the from fact, $\mathbf{F} \cdot \mathbf{n} = (x_0 + \frac{s}{2})^3$. Noting that x_0 and s are constants (relative to S_1), it follows that

$$\int\int_{S_1} \mathbf{F} \cdot \mathbf{n} dS = \int\int_{S_1} (x_0 + \frac{s}{2})^3\ dS = (x_0 + \frac{s}{2})^3 \cdot \text{vol}(S_1) = (x_0 + \frac{s}{2})^3 \cdot s^2.$$

The calculation for S_2 is similar. In this case, $\mathbf{n} = -\vec{i}$, $\mathbf{F} \cdot \mathbf{n} = -x^3$ and $-x^3$ evaluate on S_2 is the constant $-(x_0 - \frac{s}{2})^3$. Thus,

$$\int\int_{S_2} \mathbf{F} \cdot \mathbf{n} dS = \int\int_{S_2} -(x_0 - \frac{s}{2})^3\ dS = -(x_0 - \frac{s}{2})^3 \cdot \text{vol}(S_2) = -(x_0 - \frac{s}{2})^3 \cdot s^2.$$

Therefore,

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\ dS + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\ dS = (x_0+\frac{s}{2})^3\cdot s^3 + (x_0-\frac{s}{2})^3\cdot s^2$$
$$= s^2\{(x_0^3+\frac{3}{2}x_0^2 s+\frac{3}{4}x_0 s^2+\frac{s^3}{8}) - (x_0^3-\frac{3}{2}x_0^2 s+\frac{3}{4}x_0 s^2-\frac{s^3}{8})\}$$
$$= s^2\{3x_0^2 s+\frac{s^3}{4}\}$$
$$= 3x_0^2 s^3+\frac{s^5}{4}.$$

We proceedED in similar fashion for the next two faces, the right and left faces of C_s. Suppose S_3 is the right face of the cube C_s. Then $\mathbf{n} = \vec{j}$, $\mathbf{F}\cdot\mathbf{n} = 4y$, and $4y$ evaluated on S_3 equals $4(y_0+\frac{s}{2})$, a constant on S_3. Thus,

$$\int\int_{S_3} \mathbf{F}\cdot\mathbf{n}\ dS = 4(y_0+\frac{s}{2})\cdot \text{vol}(S_3) = 4(y_0+\frac{s}{2})\cdot s^2.$$

For the left face S_4, $\mathbf{n} = -\vec{j}$, $\mathbf{F}\cdot\mathbf{n} = -4y$ and $-4y$ evaluated on S_3 equals $-4(y_0-\frac{s}{2})$, a constant on S_3. Thus,

$$\int\int_{S_4} \mathbf{F}\cdot\mathbf{n}\ dS = -4(y_0-\frac{s}{2})\cdot \text{vol}(S_3) = -4(y_0-\frac{s}{2})\cdot s^2.$$

Thus,

$$\int\int_{S_3} \mathbf{F}\cdot\mathbf{n}\ dS + \int\int_{S_4} \mathbf{F}\cdot\mathbf{n}\ dS = 4(y_0+\frac{s}{2})\cdot s^2 - 4(y_0-\frac{s}{2})\cdot s^2$$
$$= 4s^3.$$

Finally, for the top face S_5, $\mathbf{n} = \vec{k}$, $\mathbf{F}\cdot\mathbf{n} = 2z^2$ and $2z^2$ on S_5 equals $2(z_0+\frac{s}{2})^2$. Thus, as before,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\ dS = 2(z_0+\frac{s}{2})^2\cdot \text{vol}(S_5) = 2(z_0+\frac{s}{2})^2\cdot s^2.$$

Similarly, if S_6 denotes the bottom face of C_s, $\mathbf{n} = -\vec{k}$, $\mathbf{F}\cdot\mathbf{n} = 2z^2$, which equals $2(z_0-\frac{s}{2})^2$ on S_6. Thus,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\ dS = -2(z_0-\frac{s}{2})^2\cdot \text{vol}(S_5) = -2(z_0-\frac{s}{2})^2\cdot s^2.$$

Therefore,

$$\int\int_{S_5} \mathbf{F}\cdot\mathbf{n}\ dS + \int\int_{S_6} \mathbf{F}\cdot\mathbf{n}\ dS = 2(z_0+\frac{s}{2})^2\cdot s^2 - 2(z_0-\frac{s}{2})^2\cdot s^2$$
$$= 4z_0 s^3.$$

Adding these six integrals, we now have

$$\int\int_{C_s} \mathbf{F}\cdot\mathbf{n}\ dS = 3x_0^2 s^3+\frac{s^5}{4}+4s^3+4z_0 s^3$$

Therefore,

$$\text{div}\ \mathbf{F}(P) = \lim_{\text{vol}(C_s)\to 0} \frac{1}{\text{vol}(C_s)} \int\int_{C_s} \mathbf{F}\cdot\mathbf{n}\ dS$$
$$= \lim_{s\to 0} \frac{1}{s^3}\cdot(3x_0^2 s^3+\frac{s^5}{4}+4s^3+4z_0 s^3)$$
$$= 3x_0^2+4+4z_0.$$

We noted that one might guess from the preceding example the following formula for the divergence in rectangular coordinate:

$$\text{div}\ \mathbf{F}(P) = \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P).$$

In cylindrical coordinates: $\mathbf{F} = (F_r, F_\theta, F_z)$ (**not** $F_r i + F_\theta j + F_z k$), and

$$\text{div } \mathbf{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}.$$

In spherical coordinates: $\mathbf{F} = (F_r \rho F_\phi, F_\theta)$, and

$$\text{div } \mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho\sin(\phi)}\frac{\partial F_\phi}{\partial \phi} + \frac{1}{\rho\sin(\phi)}\frac{\partial(\sin(\theta)F_\theta)}{\partial \theta}.$$

We then stated one of the main theorems of our course:

The Divergence Theorem. Let S be a closed (piece-wise smooth) surface that bounds the solid B in $\mathbb{R}^3$. If the first order partial derivatives of the component functions of $\mathbf{F}$ are continuous on B, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_B \text{div } \mathbf{F} \; dV.$$

We noted that the integral on the right is a standard triple integral of a scalar function over a solid in $\mathbb{R}^3$, while the integral on the left is a surface integral of the the vector field $\mathbf{F}(x, y, z)$ with respect to the *outward pointing normal* vectors. We also noted that the theorem makes intuitive sense using our coordinate free definition of the divergence of a vector field. Indeed, on the one hand, the left hand side of the equation above is the flux of $\mathbf{F}$ across the boundary of B, so that if we think of S as a chamber, full of fluid, this is the amount of fluid leaving the chamber. On the other hand, the divergence of $\mathbf{F}$ at each point $P \in B$ measures how much fluid flows from P. Summing this over all points in B should give the right hand side of the equation in the theorem.

We then showed how to use the Divergence theorem.

Example. Calculate $\int\int_S \mathbf{F} \cdot d\mathbf{S}$ with respect to the outward normal vectors, for $\mathbf{F} = xz^2 i + yx^2 j + zy^2 k$ and S the sphere of radius R centered at the origin.

Solution. We can set up the surface integral by regarding it as $\int\int_S \mathbf{F} \cdot \mathbf{n} \, dS$. For a point $P = (x, y, z)$ on S, the unit normal at P is $\frac{1}{R}(x, y, z)$. Thus $\mathbf{F} \cdot \mathbf{n}$ on S equals

$$(xz^2, yx^2, zy^2) \cdot \frac{1}{R}(x, y, z) = \frac{x^2z^2 + y^2x^2 + z^2y^2}{R}.$$

Therefore,

$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_S \mathbf{F} \cdot \mathbf{n} \; dS \\ &= \int\int_S \frac{x^2z^2 + y^2x^2 + z^2y^2}{R} \; dS, \end{aligned}$$

Which is doable, but not pleasant. Better: Use the Divergence Theorem. div $\mathbf{F} = z^2 + +x^2 + y^2$. For B the solid sphere,

$$\int\int\int_B \text{div } \mathbf{F} \; dV = \int\int\int_B x^2 + y^2 + z^2 \; dV,$$

which can easily be calculated using spherical coordinates.

$$\int\int\int_B \text{div } \mathbf{F}\ dV = \int\int\int_B x^2 + y^2 + z^2\ dV$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \cdot \rho^2 \sin(\phi)\ d\rho d\phi d\theta$$
$$= 2\pi \int_0^{\pi}\int_0^{R} \rho^4 \sin(\phi)\ d\rho d\phi$$
$$= \frac{2}{5}\pi R^5 \int_0^{\pi} \sin(\phi)\ d\phi$$
$$= \frac{4}{5}\pi R^5.$$

Therefore, $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \frac{4}{5}\pi R^5$.

We then had a heuristic discussion about why the Divergence Theorem works? We want to compare $\int\int\int_B \text{div } \mathbf{F}\ dV$ and $\int\int_S \mathbf{F} \cdot d\mathbf{S}$.

Step 1. Working in rectangular coordinates, we subdivide B into small solids solids B_i that are approximately cubes, each with volume ΔV.

Step 2. We take a point P_i in each B_i.

Step 3. div $\mathbf{F}(P_i) \approx \frac{1}{\text{vol}(B_i)} \cdot \int\int_{S_i} \mathbf{F} \cdot \mathbf{n}\ dS$, where S_i is the boundary of B_i.

Step 4. div $\mathbf{F}(P_i) \cdot \text{vol}(B_i) \approx \int\int_{S_i} \mathbf{F} \cdot \mathbf{n}\ dS$.

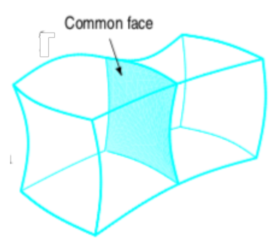
Summing the left hand side we get:

$$\sum_i \text{div } \mathbf{F}(P_i) \cdot \text{vol}(B_i) = \sum_i \text{div } \mathbf{F}(P_i)\Delta V,$$

a Reimann sum. Passing to the limit as $\Delta V \to 0$ we get:

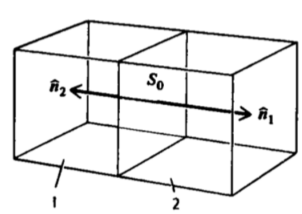
$$\int\int\int_B \text{div } \mathbf{F}\ dV.$$

Something interesting happens when we sum the terms belonging to the right hand side of Step 4.



When we integrate $\mathbf{F}$ over the boundaries of adjacent V_i, the surface integrals over common faces cancel.

Why: We integrate twice over the common face, once for S_1 and again for S_2, but once with a normal vector $\mathbf{n}$ and again with $-\mathbf{n}$. For the common face S_0:



$\int\int_{S_0} \mathbf{F}\cdot\mathbf{n_1}\, dS = \int\int_{S_0} \mathbf{F}\cdot -\mathbf{n_2}\, dS$. Thus when we add

$$\int\int_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \int\int_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS,$$

the components over S_0 cancel and the sum becomes $\int\int_{\tilde{S}} \mathbf{F}\cdot\mathbf{n}\, dS$, where $\tilde{S}$ is the combined outer shell of S_1 and S_2.

Thus, the sum of the terms in the RHS in Step 4 approximates $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, and equals it in the limit.

We ended class by mentioning that Gauss's Law is a physics version of the Divergence Theorem We did not provide the details below in class. .

Gauss' Law (Physics Version). The net electric flux through any hypothetical closed surface is equal to $\frac{1}{\epsilon 0}$ times the net electric charge q within that closed surface, where ϵ_0 is the electric constant.

Gauss' Law (Math Version). Let M be a solid in $\mathbb{R}^3$ with a smooth boundary ∂M. Assume (0,0,0) is not on the boundary ∂M. Set $\mathbf{r} = xi + yj + zk$, and $r = ||\mathbf{r}|| = \sqrt{x^2+y^2+z^2}$. Then:

$$\int\int_{\partial M} (\frac{1}{r^3}\mathbf{r})\cdot\mathbf{n}\, dS = \begin{cases} 4\pi, & if\ (0,0,0)\in M \\ 0, & if\ (0,0,0)\notin M \end{cases}.$$

In the second case, the vector field $\frac{1}{r^3}\mathbf{r}$ is defined throughout M, so that if we show its divergence is zero, then by the Divergence Theorem,

$$\int\int_{\partial M} (\frac{1}{r^3}\mathbf{r})\cdot\mathbf{n}\, dS = 0.$$

The i component of $\frac{1}{r^3}\mathbf{r}$ is $\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$. Differentiating with respect to x we get:

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3} =$$

$$(x^2+y^2+z^2)^{\frac{1}{2}}\cdot\frac{(x^2+y^2+z^2)-3x^2}{(x^2+y^2+z^2)^3} =$$

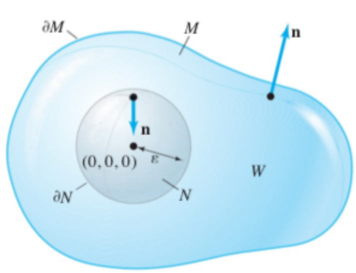
$$\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

Similarly, the corresponding partials of the j and k components are:

$$\frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \quad \text{and} \quad \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}.$$

Adding these three terms shows $\text{div}(\frac{1}{r^3}\mathbf{r}) = 0$.

Now, suppose $(0,0,0)\in M$. Let N be a sphere of radius ϵ centered at (0,0,0) contained in M. Let W be the complement of N in M. The boundary of W is $\partial N \cup \partial M$, i.e., the union of the boundaries of M and N.



Since $\text{div}\frac{1}{r^3}\mathbf{r} = 0$ over W, $\int\int_{\partial N\cup\partial M} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\, dS = 0$. Therefore,

$$\int\int_{\partial M} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\, dS = -\int\int_{\partial N} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\, dS.$$

Note that the outward normal for ∂N as part of the boundary of B is the inward normal for N. Thus,

$$\int\int_{\partial M} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\; dS = \int\int_{\partial N} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\; dS,$$

where now $\mathbf{n}$ is the outward normal for the sphere N.

On the sphere ∂N, the sphere of radius ϵ centered at the origin, $\mathbf{n} = \frac{1}{\epsilon}\mathbf{r}$ and $r = \epsilon$. Then:

$$\frac{1}{r^3}\mathbf{r}\cdot\mathbf{n} = \frac{1}{\epsilon^3}\mathbf{r}\cdot\frac{1}{\epsilon}\mathbf{r} = \frac{\epsilon^2}{\epsilon^4} = \frac{1}{\epsilon^2}.$$

Thus:

$$\begin{aligned}\int\int_{\partial M} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\; dS &= \int\int_{\partial N} \frac{1}{r^3}\mathbf{r}\cdot\mathbf{n}\; dS\\ &= \int\int_{\partial N} \frac{1}{\epsilon^2}\; dS\\ &= \frac{1}{\epsilon^2}\cdot \text{area}(\partial N) = \frac{1}{\epsilon^2}\cdot 4\pi\epsilon^2\\ &= 4\pi.\end{aligned}$$

Comments not stated in class. In rectangular coordinates, we often write $\nabla\cdot\mathbf{F}$ for div $\mathbf{F}$, since we think of ∇ as $\frac{\partial}{\partial x}\,\vec{i} + \frac{\partial}{\partial y}\,\vec{j} + \frac{\partial}{\partial z}\,\vec{k}$, which is a *differential operator*. Thus,

$$\begin{aligned}\nabla\cdot\mathbf{F} &= (\frac{\partial}{\partial x}\,\vec{i} + \frac{\partial}{\partial y}\,\vec{j} + \frac{\partial}{\partial z}\,\vec{k})\cdot(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\\ &= \text{div }\mathbf{F}.\end{aligned}$$

The use of ∇ in this way is consistent with our prior use, if we recall that the gradient of a scalar function $f(x, y, z)$ is:

$$\nabla(f) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

If we use the symbol ∂B to denote the boundary of the solid in the Divergence Theorem, and the notation $\nabla\cdot\mathbf{F}$ for div $\mathbf{F}$, then the Divergence Theorem becomes:

$$\int\int_{\partial B} \mathbf{F}\cdot d\mathbf{S} = \int\int\int_B \nabla\cdot\mathbf{F}\; dV.$$

Note how the differential operator on the domain of integration on the left hand side of the equation, moves up to the integrand on the right hand side of the equation.

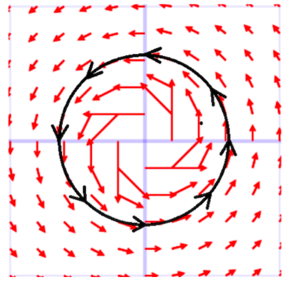
Wednesday, November 19. We began class with the discussion of Gauss's Law presented in the Daily Update from Monday, November 17. In particular, we worked through the calculation of the math version of Gaiss's law. We then turned to a dsicssuion of the curl of a vector field in $\mathbb{R}^2$.

We discussed the curl of a vector field in $\mathbb{R}^2$ and Green's Theorem.

We start with a vector field $\mathbf{F} = F_1(x, y)\vec{i} + F_2(x, y)\vec{j}$ in $\mathbb{R}^2$. If C is a closed path, the circulation of $\mathbf{F}$ along C is the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\; ds,$$

where we assume C is oriented in a counter-clockwise direction.



Throughout we assume that the first order partial derivatives of the component functions of $\mathbf{F}$ exist and are continuous. Later we will need to assume that all second order partial derivatives of the components functions exist and are continuous.

Definition. The curl of $\mathbf{F}$ at the point P is the circulation per unit area at the point P and is given by the formula

$$(\text{Curl }\mathbf{F})(P) = \lim_{\text{area}(D)\to 0} \frac{1}{\text{area}(D)} \int_{\partial D} \mathbf{F}\cdot d\mathbf{r},$$

where D is a region in $\mathbb{R}^2$ whose boundary ∂D is a simple closed curve. A simple closed curve is a (piecewise) smooth curve with no self intersections. We have the following properties of the curl of $\mathbf{F}$:

(i) Curl $\mathbf{F}$ is a scalar function.
(ii) We can calculate $(\text{Curl }\mathbf{F})(P)$ using shrinking regions of our choice.
(iii) The value of $(\text{Curl }\mathbf{F})(P)$ does not depend upon the regions we choose.
(iv) $(\text{Curl }\mathbf{F}\ )(P)$ is also independent of the coordinate system.

Example. Calculate $(\text{Curl }\mathbf{F})(P)$ for $\mathbf{F} = -yi + xj$ and $P = (x_0, y_0)$, using shrinking disks of radius epsilon.

Solution. If we let D_ϵ denote the disk of radius ϵ centered at P, and C_ϵ denote its boundary, then using the parametrization of $C_\epsilon : \mathbf{r}(t) = (\epsilon\cos(t) + x_0, \epsilon\sin(t) + y_0)$, with $0 \le t \le \pi$, we have

$$\begin{aligned}
\text{Curl }\mathbf{F} &= \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2} \int_{C_\epsilon} \mathbf{F}\cdot\ d\mathbf{r}\\
&= \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2} \int_0^{2\pi} (-\epsilon\sin(t) - y_0, \epsilon\cos(t) + x_0)\cdot(-\epsilon\sin(t), \epsilon\cos(t))\ dt\\
&= \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2} \int_0^{2\pi} \epsilon^2 + \epsilon y_0\sin(t) + \epsilon x_0\cos(t)\ dt\\
&= \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2} \{\epsilon^2 t - \epsilon y_0\cos(t) + \epsilon x_0\sin(t)\}\Big|_0^{2\pi}\\
&= \lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2}\cdot 2\pi\epsilon^2\\
&= 2.
\end{aligned}$$

How do we calculate the curl using rectangular coordinates? Answer:

$$\text{Curl }\mathbf{F}(P) = \frac{\partial F_2}{\partial x}(P) - \frac{\partial F_1}{\partial y}(P).$$

Checking the formula with Example 1: $\mathbf{F} = -y\vec{i} + xj$. $\frac{\partial F_2}{\partial x} = \frac{\partial x}{\partial x} = 1$. $\frac{\partial F_1}{\partial y} = \frac{\partial -y}{\partial y} = -1$.

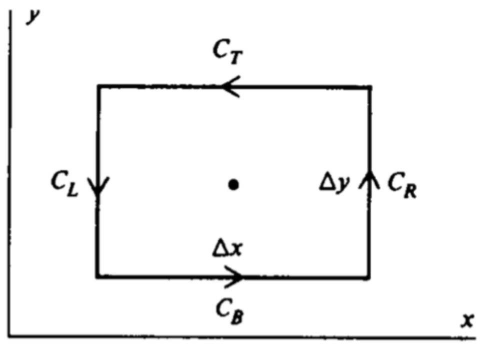
Solution.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2 = \text{Curl }\mathbf{F}.$$

Thursday, November 20. After recalling the limit definition of Curl $\mathbf{F}$, we can an informal explanation of why in rectangular coordinates one has

$$\text{Curl }\mathbf{F}(P) = \frac{\partial F_2}{\partial x}(P) - \frac{\partial F_1}{\partial y}(P).$$

We start with a $(\Delta x) \times (\Delta y)$ square centered at (x_0, y_0). We travel counter-clockwise around the square.



Along the base C_B, the unit tangent is $\vec{i}$, so that $\mathbf{F} \cdot \mathbf{t}$ is F_1 along C_B. Since $y = y_0 - \frac{\Delta y}{2}$ along the base, $\mathbf{F} \cdot \mathbf{t} = F_1(x, y_0 - \frac{\Delta y}{2})$. Moreover, if Δx is small, and x_0 is the center of C_B, then the average value of F_1 along C_B is approximately $F_1(x_0, y_0 - \frac{\Delta y}{2})$. Thus,

$$\int_{C_B} \mathbf{F} \cdot \mathbf{t}\, ds = \int_{C_B} F_1\, ds \approx F_1(x_0, y_0 - \frac{\Delta y}{2})\Delta x. \text{ (Avg value times length)}$$

Along the top C_T, the unit tangent is $-\vec{i}$, so that $\mathbf{F} \cdot \mathbf{t}$ is -F_1 along C_T. Thus, similarly, we have

$$\int_{C_T} \mathbf{F} \cdot \mathbf{t}\, ds = \int_{C_B} -F_1\, ds \approx -F_1(x_0, y_0 + \frac{\Delta y}{2})\Delta x. \text{ (Avg value times length)}$$

Adding these terms we have

$$\int_{C_B + C_T} \mathbf{F} \cdot \mathbf{t}\, ds \approx -\Delta x \cdot \{F_1(x_0, y_0 + \frac{\Delta y}{2}) - F_1(x_0, y - \frac{\Delta y}{2})$$

Dividing by the area $\Delta x \Delta y$, we get:

$$\frac{1}{\Delta x \Delta y} \int_{C_B + C_T} \mathbf{F} \cdot \mathbf{t}\, ds = -\frac{1}{\Delta y} \cdot \{F_1(x_0, y_0 + \frac{\Delta y}{2}) - F_1(x_0, y - \frac{\Delta y}{2})\}.$$

Taking the limit as $\Delta x, \Delta y \to 0$ gives

$$\lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_{C_B + C_T} \mathbf{F} \cdot \mathbf{t}\, ds = -\frac{\partial F_1}{\partial y}(x_0, y_0).$$

A similar analysis yields

$$\lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_{C_R + C_L} \mathbf{F} \cdot \mathbf{t}\, ds = \frac{\partial F_2}{\partial x}(x_0, y_0).$$

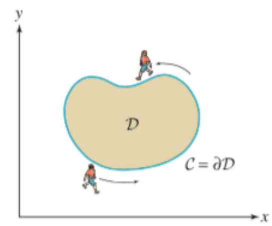
Adding we get

$$\begin{aligned} (\text{Curl } \mathbf{F})(x_0, y_0) &= \lim_{\Delta, x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_C \mathbf{F} \cdot \mathbf{t}\, ds \\ &= \frac{\partial F_2}{\partial x}(x_0, y_0) - \frac{\partial F_1}{\partial y}(x_0, y_0). \end{aligned}$$

We then discussed at length another major theorem from vector calculus.

Green's Theorem. Let $D \subseteq \mathbb{R}^2$ be a domain whose boundary ∂D is a simple closed curve, oriented counterclockwise. Suppose further that the component functions of $\mathbf{F}$ have continuous first partial derivatives. Then:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int \int_D \text{Curl } \mathbf{F}\, dA.$$



This was followed by:

Example 2. Verify Green's Theorem for D the disk $0 \leq x^2 + y^2 \leq R^2$ and $\mathbf{F} = (x-y)\vec{i} + (x+y)\vec{j}$.

Solution. Let C denote the circle of radius R centered at the origin, so that $C = \partial D$. Note that we take the usual parametrization of C, which gives the correct orientation. Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= \int_0^{2\pi} (R\cos(t) - R\sin(t), R\cos(t) + R\sin(t)) \cdot (-R\sin(t), R\cos(t))\ dt \\ &= \int_0^{2\pi} R^2\ dt \\ &= 2\pi R^2.\end{aligned}$$

On the other hand, Curl $\mathbf{F} = \frac{\partial(x+y)}{\partial x} - \frac{\partial(x-y)}{\partial y} = 1 - (-1) = 2$. Thus,

$$\begin{aligned}\int\int_D \text{Curl}\mathbf{F}\ dA &= \int\int_D 2\ dA \\ &= 2 \cdot \text{area}(D) \\ &= 2\pi R^2,\end{aligned}$$

as expected.

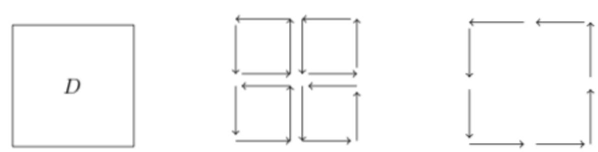
When then gave a heuristic proof of Green's Theorem. We start with $\int\int_D \text{Curl } \mathbf{F}\ dA$.

Step 1. Subdivide D into small rectangular regions D_i, and select $P_i \in D_i$.

Step 2. $\int\int_D \text{Curl } \mathbf{F}\ dA \approx \sum_i \text{Curl } \mathbf{F}(P_i)\text{Area}(D_i)$.

Step 3. $\sum_i \text{Curl } \mathbf{F}(P_i) \cdot \text{Area}(D_i) \approx \sum_i (\frac{1}{\text{area}(D_i)} \int_{\partial D_i} \mathbf{F} \cdot d\mathbf{r}) \cdot \text{area}(D_i) \approx \sum_i \int_{\partial D_i} \mathbf{F} \cdot d\mathbf{r}$.

Step 4. The line integrals along common boundaries cancel, since the tangent vectors point in opposite directions,



to get $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$.

Example 3. This example shows how to apply Green's theorem. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle centered at (1,5) and $\mathbf{F} = (x^5 + 3y)i + (2x - e^{y^3})j$.

Solution. Using the standard parametrization $\mathbf{r}(t) = (\cos(t) + 1), \sin(t) + 5)$, $0 \leq t \leq 2\pi$ leads to

$$\int_C \mathbf{F} \cdot d\mathbf{r} =$$

$$= \int_0^{2\pi} ((\cos(t)+1)^5 + 3(\sin(t)+5))(-\sin(t)) + (2(\cos(t)+1) - e^{(\sin(t)+5)^3})(\cos(t))\ dt,$$

which can actually be done with a substitution. Better: Use Green's Theorem. Curl $\mathbf{F}$ = 2-3 = -1, therefore,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D \text{Curl } \mathbf{F}\ dA = \int\int_D -1\ dA = -\text{area}(D) = -\pi.$$

Comments on Green's Theorem.

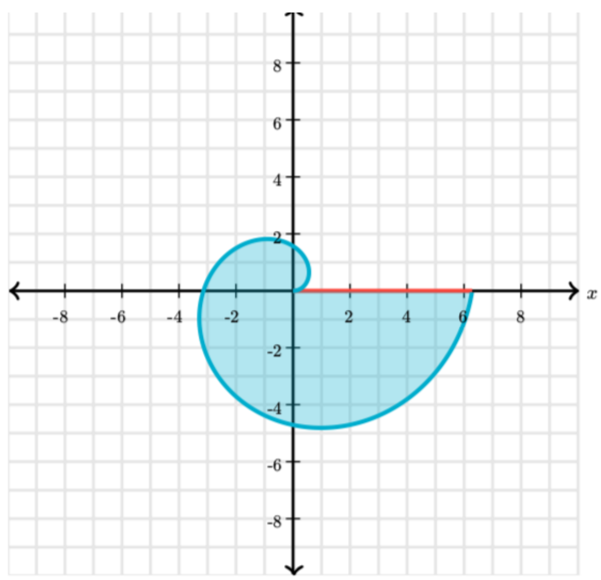
(i) Other notation for $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\int_C F_1(x,y)\ dx + F_2(x,y)\ dy$, so that

$$\int\int_D \text{Curl } \mathbf{F}\ dA = \int_C F_1(x,y)\ dx + \int_C F_2(x,y)\ dy.$$

(ii) The area of the domain D can be calculated as a line integral. Set $\mathbf{F} = -\frac{y}{2}i + \frac{x}{2}j$. So, Curl $\mathbf{F} = 1$.

$$\begin{aligned}\text{area}(D) &= \int\int_D 1\ dA \\ &= \int\int_D \text{Curl } \mathbf{F}\ dA \\ &= \int_C \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

Example 4. Find the area enclosed by the spiral $C : \mathbf{r}(t) = (t\cos(t), t\sin(t))$, with $0 \le t \le 2\pi$ and the part of the x-axis shown below.



Solution. Note that we need to include the line segment $L : \mathbf{L}(t) = 2\pi(1-t)\vec{i} + 0\vec{j}$, $0 \le t \le 1$, since the spiral itself is not a closed curve, while $C \cup L$ together give a closed curve, with positive orientation.

Over the spiral we have:

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= (-\frac{1}{2}t\sin(t), \frac{1}{2}t\cos(t))\cdot(-t\sin(t)+\cos(t), t\cos(t)+\sin(t)). \\ &= \frac{1}{2}t^2\sin^2(t) - \frac{1}{2}t\sin(t)\cos(t) + \frac{1}{2}t^2\cos(t) + \frac{1}{2}t\cos(t)\sin(t) \\ &= \frac{1}{2}t^2.\end{aligned}$$

Thus:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{1}{2}t^2\ dt = \frac{1}{2}\frac{(2\pi)^3}{3} = \frac{4\pi^3}{3}.$$

For the line segment: $\mathbf{L}(t) = ((1-t)2\pi, 0)$, with $0 \le t \le 1$. Note that $\mathbf{L}'(t) = (-1, 0)$.

$$\mathbf{F}(\mathbf{L}(t))\cdot\mathbf{L}'(t) = (0, (1-t)\pi)\cdot(-1,0) = 0.$$

Therefore:

$$\int_L \mathbf{F}\cdot d\mathbf{r} = 0.$$

Consequently:

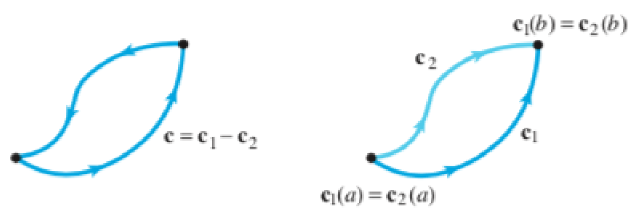
$$\begin{aligned}\text{area enclosed by spiral} &= \int_{C\cup L}\mathbf{F}\cdot d\mathbf{r}\\ &= \int_C \mathbf{F}\cdot d\mathbf{r} + \int_L \mathbf{F}\cdot d\mathbf{r}\\ &= \frac{4\pi^3}{3} + 0 = \frac{4\pi^3}{3}.\end{aligned}$$

Friday, November 21. We began class with a discussion of path independence of line integral as an application of Green's Theorem, starting with the following example.

Definition. A vector field $\mathbf{F}$ is a conservative vector field if for any two points $P, Q \in \mathbb{R}^2$, and any two curves C_1, C_2 connecting P, Q,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

When does this occur?



If

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_{C_2}\mathbf{F}\cdot d\mathbf{r}.$$

Then

$$0 = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_{-C_2}\mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int\int_D \frac{dF_2}{dx} - \frac{\partial F_1}{\partial y}\ dA.$$

This suggests $\frac{dF_2}{dx} - \frac{\partial F_1}{\partial y} = 0$, leading to the following theorem.

Theorem. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$, such that the first and second order partials of the component functions of $\mathbf{F}$ are continuous.. The following are equivalent:

(i) $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, for C_1, C_2 traveling from P and Q.
(ii) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, for simple closed curves C.
(iii) Curl $\mathbf{F} = 0$.
(iv) $\mathbf{F} = \nabla f(x, y)$, for some scalar function $f(x, y)$.

Moreover, in this case, if C is any smooth curve from P to Q in $\mathbb{R}^2$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla F \cdot d\mathbf{r} = f(Q) - f(P),$$

which is a generalization of the fundamental theorem of calculus. We also noted that the condition of second order partial guarantees equality of mixed second order partial derivatives, from with it easily follows that if $\mathbf{F} = \nabla f$, then Curl $\mathbf{F} = 0$.

Example from November 14 revisited. Set $\mathbf{F} = (3y + 1)\vec{i} + 3x\vec{j}$. Then, $\mathbf{F} = \nabla f(x, y)$, for the scalar function $f(x, y) = 3xy + x$. Thus, $\mathbf{F}$ is a conservative vector field. Moreover, we saw

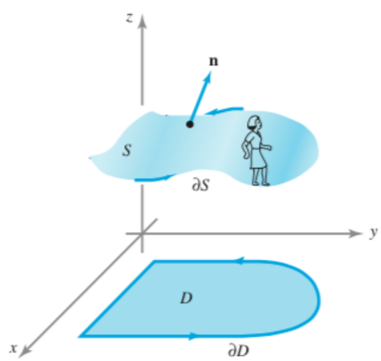
$$f(Q) - f(P) = f(0, 1) - f(1, 0) = 0 - 1 \ = -1 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{r}.$$

The notion of the curl of a vector field in $\mathbb{R}^3$ can be defined similarly, as in the previous lecture, however in this case, Curl $\mathbf{F}$ is a vector field. The definition gives the values of the component of the curl in a given direction.

Definition. Given a vector field $\mathbf{F}(x, y, z)$ defined in a region of $\mathbb{R}^3$, a point $P \in \mathbb{R}^3$, and a unit normal vector $\mathbf{n}$, the component of the Curl of $\mathbf{F}(x, y, z)$ at P in the direction perpendicular to $\mathbf{n}$ is defined as follows:

$$\text{Curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot d\mathbf{r},$$

where the limit is taken over closed curves C such that $\mathbf{n}$ at P is normal to S as the areas ΔS enclosed by those curves tend to 0. In this definition, the curve should be oriented according to the right hand thumb rule.



Comments. Given a vector field, $\mathbf{F}$:

(i) This definition is independent of the coordinate system.
(ii) If we choose a coordinate system, say rectangular coordinates, then taking $\mathbf{n}$ to be $\vec{i}$, then $\vec{j}$, then $\vec{k}$, we get components of the vector field **Curl F** at the point P. Thus, **Curl F** is a vector field derived from the field $\mathbf{F}$.
(iii) The limit

$$\textbf{Curl } \mathbf{F}(P) \cdot \mathbf{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \int_C \mathbf{F} \cdot d\mathbf{r}$$

gives the circulation of $\mathbf{F}$ at P per unit area about an axis in the direction of $\mathbf{n}$.

(iv) If we take $\mathbf{n} = \vec{k}$, and assume P lies in the xy-plane, we get the curl of $\mathbf{F}$ at P as defined in the previous lecture.

A similar, though more complicated, argument as the one given in the previous lecture gives

Formula for Curl in $\mathbb{R}^3$: $\mathbf{F} = F_1 i + F_2 j + F_3 k$, then

$$\textbf{Curl F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})\vec{i} + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\vec{j} + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\vec{k}.$$

We noted that one can also use the notation $\nabla \times \mathbf{F}$ to denote **Curl F**.

In cylindrical coordinates: $\mathbf{F} = (F_r, F_\theta, F_z)$ and

$$\begin{aligned} (\nabla \times \mathbf{F})_r &= \frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \\ (\nabla \times \mathbf{F})_\theta &= \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \\ (\nabla \times \mathbf{F})_z &= \frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta} \end{aligned}$$

In spherical coordinates: $\mathbf{F} = (F_\rho, F_\phi, F_\theta)$, and

$$\begin{aligned} (\nabla \times \mathbf{F})_\rho &= \frac{1}{\rho \sin(\phi)}\frac{\partial(\sin(\phi)F_\theta)}{\partial \phi} - \frac{1}{\rho \sin(\phi)}\frac{\partial F_\phi}{\partial \theta} \\ (\nabla \times \mathbf{F})_\phi &= \frac{1}{\rho \sin(\phi)}\frac{\partial(\rho F_\rho)}{\partial \theta} - \frac{1}{\rho}\frac{\partial(\rho F_\theta)}{\partial \rho} \\ (\nabla \times \mathbf{F})_\theta &= \frac{1}{\rho}\frac{\partial(\rho F_\theta)}{\partial \rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi} \end{aligned}$$

We then began a discussion of the last major theorem of the semester, one of the most important theorems in vector calculus.

Stoke's Theorem. Let $S \subseteq \mathbb{R}^3$ be a smooth oriented surface with whose boundary ∂S is a simple closed curve, oriented according to the right hand thumb rule. If $\mathbf{F}$ is a vector field whose components have continuous first order and second order partial derivatives, then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S \textbf{Curl F} \cdot d\mathbf{S} = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Moreover: If S is a closed surface, $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$.

We first noted that we could derive a heuristic proof of Stokes theorem along the lines of the argument given for Green's theorem given in the previous lecture by using the intrinsic definition of the curl and the fact that one can partition S so that interior line integrals along common boundaries in the partition cancel. We also gave three explanations as to why $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$, when S is a closed surface: (i) S has no boundary, so using Stoke's theorem, the line integral is zero; (ii) Using the divergence theorem, $\int\int_S \nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int\int_B \nabla \cdot (\nabla \times \mathbf{F})\ dV$, which is zero, since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, under suitable differentiability conditions; (iii) Thinking of S as the union of two open surfaces $S = S_1 \cup S_2$ joined along a common boundary, and noting that $\partial S_2 = -\partial S_1$, so that

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} + \int\int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-\partial S_1} \mathbf{F} \cdot d\mathbf{r} = 0.$$

We ended class by working problem 2 on the worksheet from November 18, asking to verify Stokes theorem, obtaining 18π for each expression in the theorem.

Monday, November 14. We began class by restating Stoke's Theorem and discussed the *nabla notation* introduced in the Daily Update of November 17, $\nabla = \frac{\partial}{\partial x}\ \vec{i} + \frac{\partial}{\partial y}\ \vec{j} + \frac{\partial}{\partial z}\ \vec{k}$, which is a *differential operator*, noting that the Divergence Theorem with this notation becomes $\int\int\int_B \nabla \cdot \mathbf{F}\ dV = \int\int_{\partial B} \mathbf{F} \cdot d\mathbf{S}$, where ∂B is the closed boundary of the solid B and Stokees Theorem becomes $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$, under the previously stated conditions for each theorem.

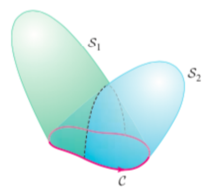
This led to a brief discussion of the following theorem, which can be considered a three variable analog of the theorem characterizing conservative vector fields.

Theorem. For a vector field $\mathbf{F}$ on $\mathbb{R}^3$ whose component functions have continuous first and second order partials, the following statements are equivalent.

(i) $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$.

(ii) $\int\int_S \mathbf{F} \cdot d\mathbf{S} = 0$, for all closed surfaces S.

(iii) $\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} \mathbf{F} \cdot d\mathbf{S}$ for any two open surfaces S_1 and S_2 sharing a common boundary with the same orientation.



Thus, (iii) is a surface integral analogue of the path independence of line integrals of conservative vector fields that we saw above.

We then illustrated the latter point further by considering $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where S is the surface represented by the graph of $z = e^{-(x^2+y^2)}$, with $z \geq \frac{1}{e}$ and $\mathbf{F} = (e^{y+z} - 2y)\vec{i} + (xe^{y+z} + y)\vec{j} + e^{x+y}\vec{k}$. We saw that computing $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ and $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ directly lead to unmanageable double and single integrals. However, since $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ is independent of surfaces sharing the same boundary, we saw that the calculation is much easier if we take S' the disk of radius 1 at the $z = \frac{1}{e}$ level, so that $\partial S = \partial S'$, and thus, $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_{S'} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 2\pi$. (Note: When $z = \frac{1}{e}$, we get $e^{-1} = e^{-(x^2+y^2)}$, so $1 = x^2 + y^2$.)

Though not stated in class, we can end the collection of Daily Updates with the following two Observations.

1. Using the symbol ∇, the Divergence Theorem and Stoke's Theorem become:

$$\int\int_{\partial B} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_B \nabla \cdot \mathbf{F}\ dV$$
$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Note how the differential operator on the domain of integration on the left hand side of the equations, moves up to the integrand on the right hand side of the equations. There are more general theorem of this type when one studies a much more theoretical course on the calculus of several variables in arbitrary dimension.

2. Just like areas can be calculated using line integrals via Green's Theorem, we can calculate volumes as surface integrals, using the divergence theorem. Taking B to be a closed and bounded solid in $\mathbb{R}^3$ whose boundary ∂B is a smooth closed surface, if we set $\mathbf{F} := \frac{1}{3} \cdot (x\vec{i} + y\vec{j} + z\vec{k})$, then

$$\text{volume}(B) = \int\int\int_B 1\ dV = \int\int\int_B \nabla \cdot \mathbf{F}\ dV = \int\int_{\partial B} \mathbf{F} \cdot d\mathbf{S}.$$

Monday, December. Jake presented review slides for the final exam.

Wednesday, December 3. The class worked on practice problems for the final exam.